MATH 20
Supplementary Materials

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GEOMETRY REVIEW

A. COMPLEMENTARY & SUPPLEMENTARY ANGLES

If the measures of angle A and B are positive then:
a) A and B are said to be complementary if their sum is 90°,
b) A and B are said to be supplementary if their sum is 180°.

Example: If A = 25°, then a) name its complement;

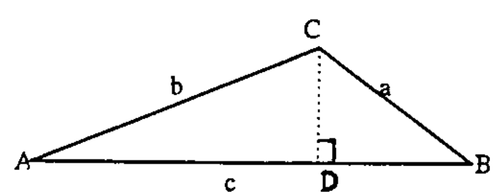
b) name its supplement.

a) The complement is 65° since 90° - 25° = 65° (or 25° + 65° = 90°).

b) The supplement is 155° since 180° - 25° = 155° (or 25° + 155° = 180°).

B. TRIANGLES

A triangle is comprised of three sides and three angles. The angles are referred to as the vertices of the triangle. It is common practice to label the vertices with capital letters; A, B, C and the sides opposite each vertex with the corresponding small letters: a, b, c. The dotted line CD is called the altitude of Δ ABC. See the example below.

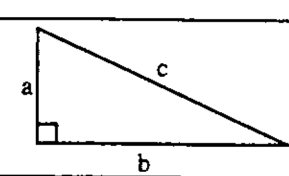


It is important to realize the sum of the three vertices is 180°.

$m\angle A + m\angle B + m\angle C = 180°$ or the sum of the measure of the three vertices is 180°.

C. PYTHAGOREAN THEOREM

The sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse:
$$a^2 + b^2 = c^2$$

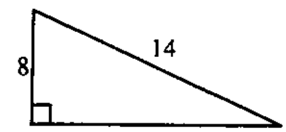


The converse of the Pythagorean Theorem is:

If the sum of the squares of the lengths of two sides of a triangle is equal to the square of the length of the third side, then the triangle is a right triangle.

Example 1:

If ΔABC is a right triangle , find the length of the missing leg.



$$8^2 + x^2 = 14^2$$
$$64 + x^2 = 196$$
$$x^2 = 132$$
$$x = \sqrt{132} \approx 11.49$$

Example 2:

Can 15, 8, 17 be the lengths of sides of a right triangle?

Is $8^2 + 15^2 = 17^2$?

$64 + 225 = 289$

$289 = 289$ Hence the lengths would form a right triangle.

D. TYPES TRIANGLES:

1) equilateral -- all the sides are of equal length and all the angles are equal. all angles are 60°

2) isosceles -- two sides have the same length
the angles opposite the equal sides are also equal

3) acute -- all angles are less than or equal to 90°

4) obtuse -- one angle has a measure greater than 90°

5) right -- one angle is exactly 90°

6) isosceles right -- the angles must measure 45° - 45° - 90°

7) 30°, 60°, 90°

The last two cases; 6 and 7, are considered **special right triangles**.

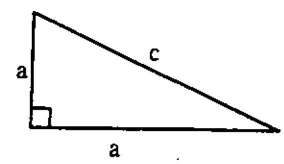
E. SPECIAL RIGHT TRIANGLES:

Theorem

In an isosceles right triangle, the hypotenuse is $\sqrt{2}$ times the length of a leg.

Proof: Let "a" be the legs of the right triangle and c the hypotenuse.

$$a^2 + a^2 = c^2$$
$$2a^2 = c^2$$
hence $\sqrt{2}\,a = c$.

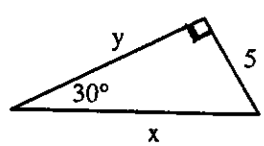


Theorem

In a 30° - 60° - 90° triangle, the length of the leg opposite the 30° angle is half the length of the hypotenuse, and the length of the leg opposite the 60° angle is $\sqrt{3}$ times the length of the other leg.

Example 1:

Find the values for x and y.



Since the side opposite the 30° is 5, then the hypotenuse, x, must be twice five.

That is x = 2 (5) = 10.

The third angle must be 60°, hence y must be $\sqrt{3}$ times the short leg.

Therefore, $y = \sqrt{3}\,(5) = 5\sqrt{3}$.

F. SIMILAR TRIANGLES

When two geometric figures have the same "shape" but are possibly of different "sizes", they are said to be **similar**. Two figures are similar if one is a scale drawing, or scale model, of the other. In other words, one figure is an "enlargement" (or "reduction") of the other. The symbol for "is similar to" is ~ . When two figures have the same shape **and** all parts have equal measures, the two figures are said to be **congruent**. (≅)

Definition

Two polygons are **similar** if and only if the following two conditions are satisfied:

1. All pairs of corresponding angles are congruent.
2. All pairs of corresponding sides are proportional.

Postulate

If the three angles of one triangle are congruent to the three angles of a second triangle, then the triangles are similar (AAA).

Corollary

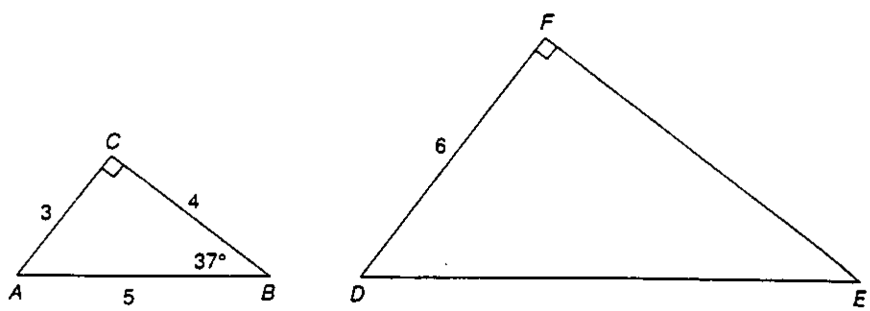
If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar (AA).

Corollary

If in Δ ABC and Δ A'B'C' we have $\frac{AB}{A'B'} = \frac{AC}{A'C'}$ and $m\angle A = m\angle A'$, then $\Delta ABC \sim \Delta A'B'C'$. In words: If we have a pair of sides of Δ ABC proportional to a corresponding pair of sides of Δ A'B'C', and the **included** angles are equal, then the two triangles are similar.

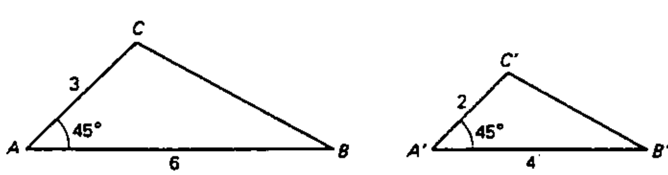
Example 1

If $\Delta ABC \sim \Delta DEF$ in the figure below, use the indicated measures to find the measures of the remaining parts of each of the triangles.



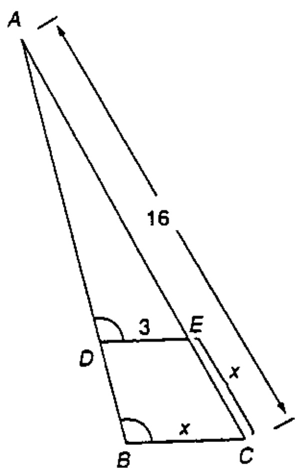
Example 2

In the picture below, $\Delta ABC \sim \Delta A'B'C'$ since $\frac{AB}{A'B'} = \frac{6}{4} = \frac{AC}{A'C'}$ and $m\angle A = 45° = m\angle A'$



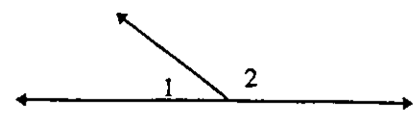
Example 3

$\angle ADE \cong \angle B$ in the figure below. If $DE = 3$, $AC = 16$, and $EC = BC$, find the length BC.



GEOMETRY REVIEW -- Homework assignment

1. In the figure at the right, the measure of $\angle 1$ is 9° less than half the measure of $\angle 2$. Find the measure of $\angle 1$ and $\angle 2$.



2. For each of the following name its complement and supplement, if possible.

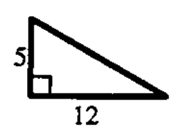
   a) 20° ________ ________

   b) 63° ________ ________

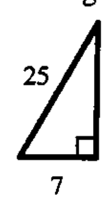
   c) 110° ________ ________

3. For each of the following find the length of the missing side.

   a)

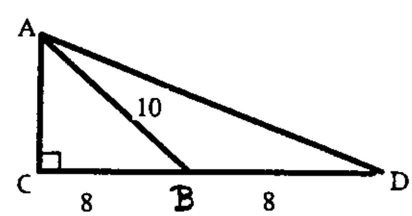


   b)



4. Find the length of the altitude of a triangle with all sides of length 10.

5. For the drawing at the right, find AD.



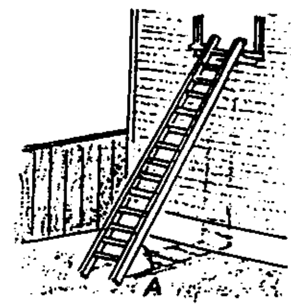
6. Which of the following represents lengths of sides of a right triangle?

   a) 4, 5, 9

   b) 9, 40, 41

   c) 20, 48, 52

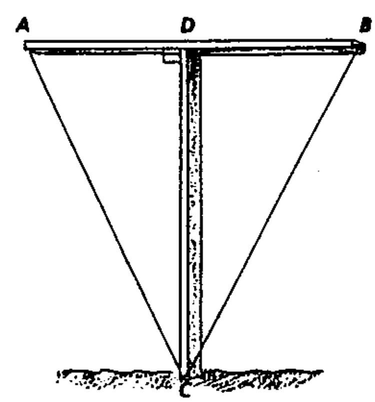
7. If the top edge of a 15 foot ladder reaches a window sill 12 feet above the ground, how far from the house is the ladder placed?



8. For maximum safety the distance between the base of a ladder and building should be one-third of the length of the ladder. If a window is 20 feet above the ground, how long a ladder is needed to meet the safety condition?

9. Find the length of a side of an equilateral triangle if its altitude is six inches long.

10. The frame for this trellis is made with two pieces of wood each 1.5 meters long. Wire is strung from the ends of the crossbar to the tip of the vertical piece of the frame. How much wire is needed in the complete frame for AC and BC?



11. Identify the type of triangle formed by the following sets of lengths of sides (ie. equilateral, isosceles, isosceles right, right, 30° - 60° -90°, no triangle, etc).

a) $8,\ 8,\ 8\sqrt{2}$

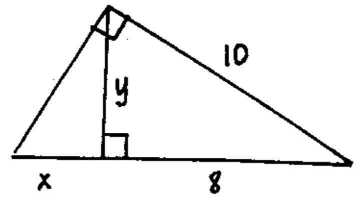
b) $\sqrt{2},\ \sqrt{3},\ \sqrt{5}$

c) $8,\ 8\sqrt{3},\ 16$

d) 8, 7, 16

e) 6, 6, 10

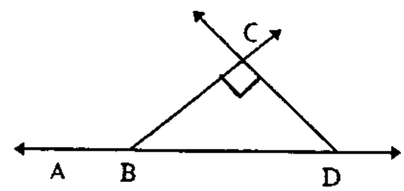
12. Find x and y.



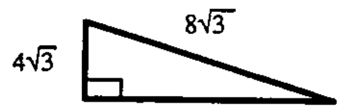
13. For the figure at the right, what angle is:

a) complementary to ∠CBD?

b) supplementary to ∠CBD?



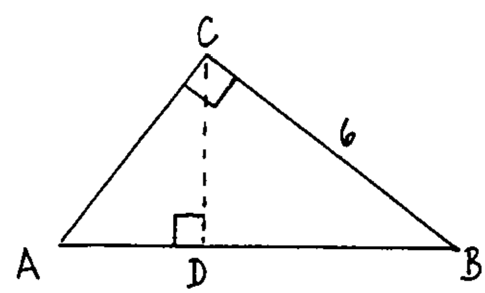
14. Find the missing side of the triangle:



15. Given the isosceles right triangle ACB with sides of length 6.

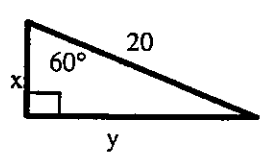
a) Find the altitude CD.

b) Find the area of triangle ACB.

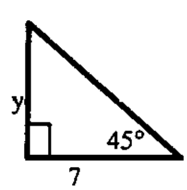


16. Find x or y or both. Problem c) is an equilateral triangle.

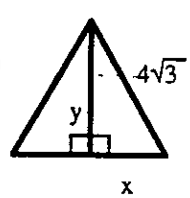
a)



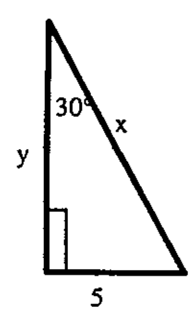
b)



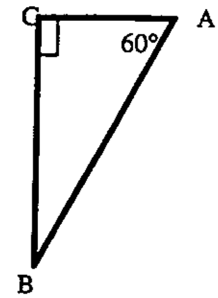
c)



d)



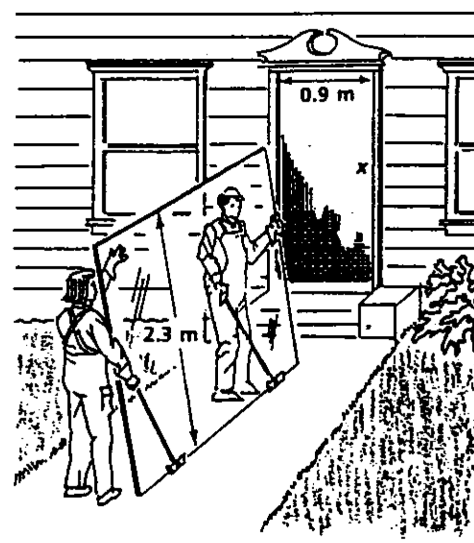
17. Complete each of the following:



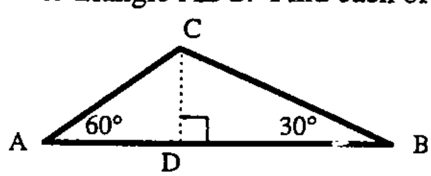
a) $AC = 6\sqrt{6}$; $BC =$ ________

b) $BC = 12$; $AB =$ ________

18. The glass for the picture window is 2.3 meters wide. The doorway is 0.9 meters wide. About how high must the doorway be in order for the contractor's helpers to get the glass through the doorway?

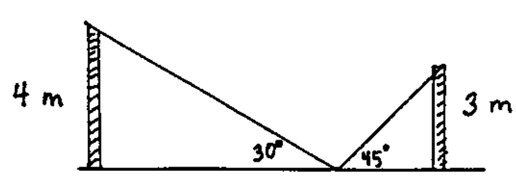


19. Consider the triangle ABC. Find each of the following:

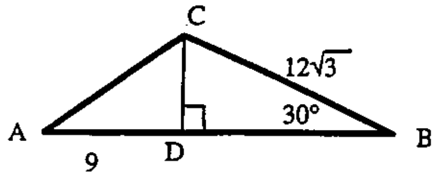


AB	BC	CD	AD	DB	AC
8	________	________	________	________	________
________	2	________	________	________	________
________	________	________	________	$10\sqrt{3}$	________

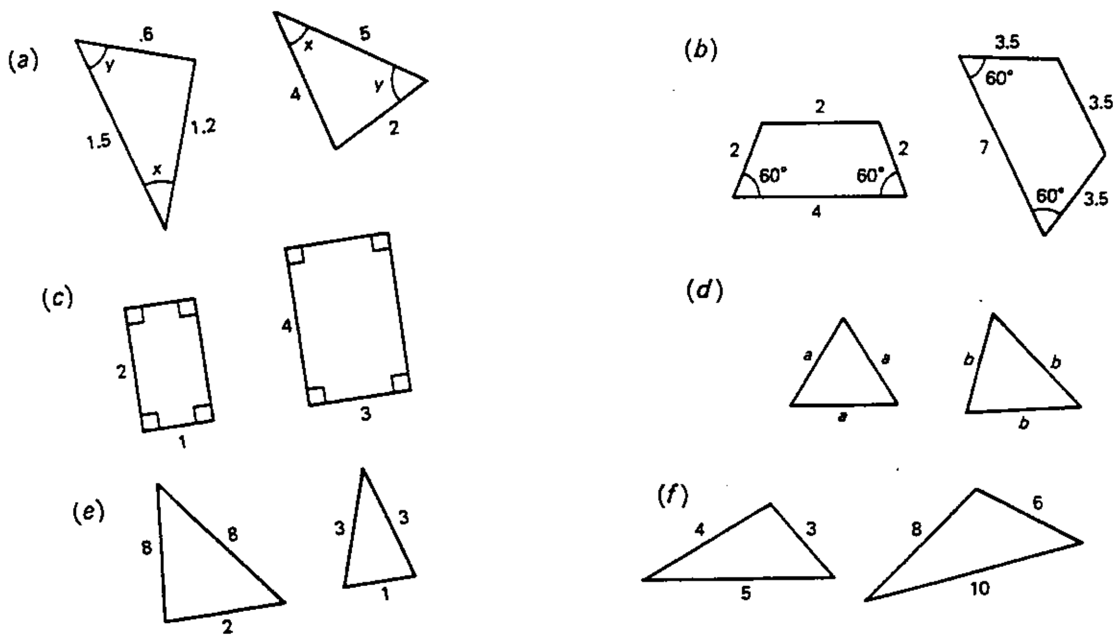
20. Find the total length of the two wire braces for these two poles.



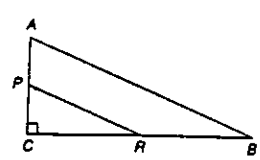
21. Find the perimeter for the following triangle ABC.



22. Without using drawings, decide which of the following pairs of polygons are similar. If you think the polygons are similar, show how they satisfy conditions 1. and 2. of the definition. (Caution: Some of the figures have not been drawn accurately!)



23. Given: $\Delta ABC \sim \Delta PRC$, $m\angle A = 67°$, PC = 5, CR = 12, PR = 13, and AB = 26.

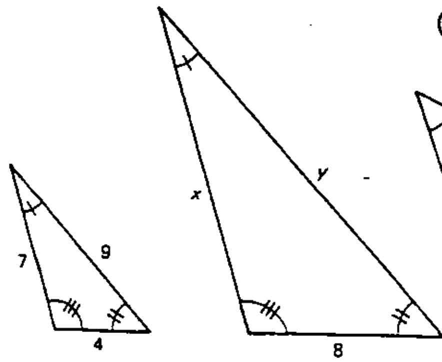


Find:

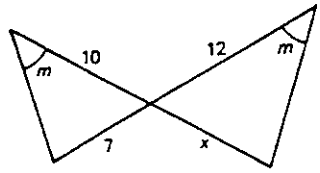
(a) $m\angle B$ (b) $m\angle RPC$ (c) AC (d) CB

24. In each case, find both x and y, or just x in those cases where only x appears. Explain your answers.

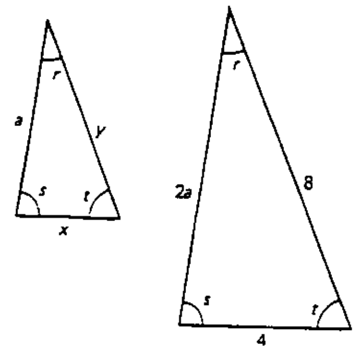
(a)



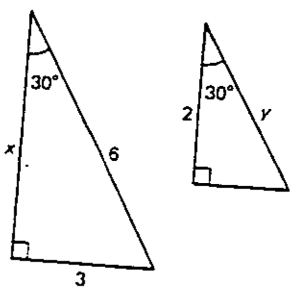
(b)



(c)



(d)



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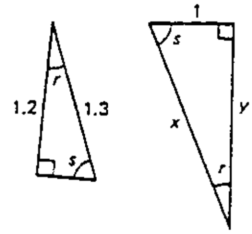
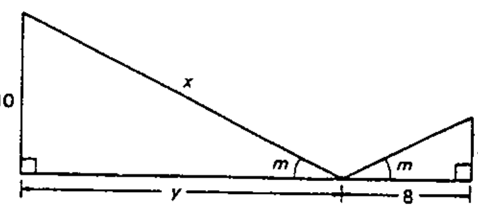
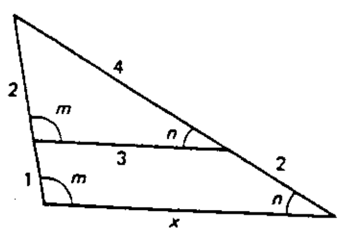


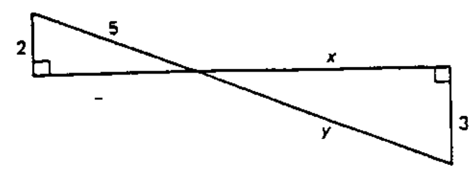
(f)



(g)

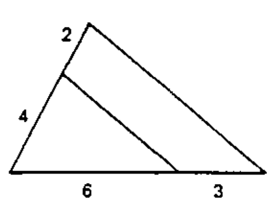


(h)

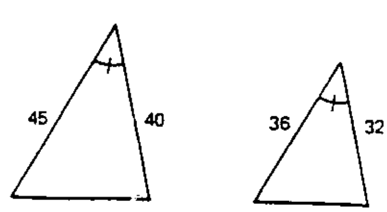


25. In each case, explain why the two triangles must be similar.

(a)



(b)



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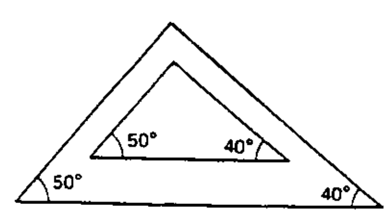
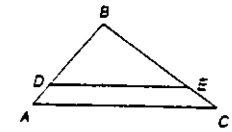


In exercises 26-29, $\Delta ABC \sim \Delta DBE$ in the accompanying drawing.



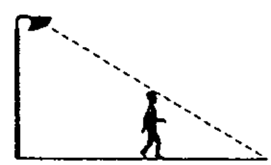
26. Given: AC = 8, DE = 6, CB = 6
Find: EB

27. Given: AC = 10, CB = 12, E is the midpoint of CB
Find: DE

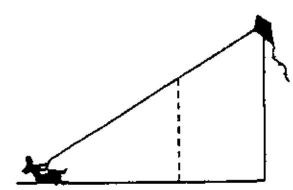
28. Given: AC = 10, DE = 8, AD = 4
Find: DB

29. Given: CB = 12, CE = 4, AD = 5
Find: DB

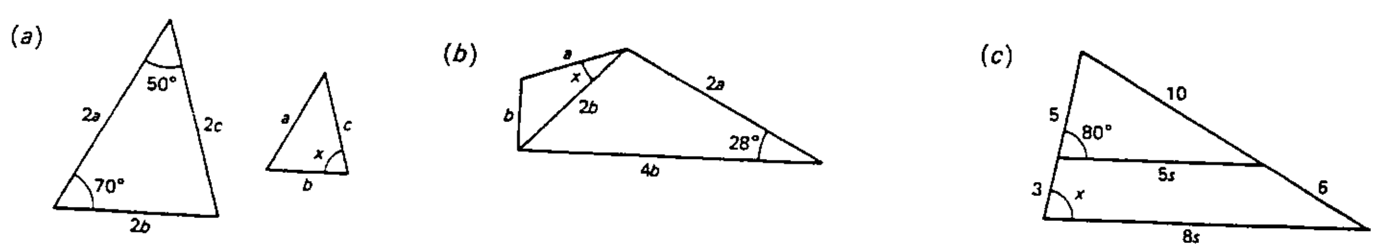
30. A person who is walking away from a ten-foot lamppost casts a shadow six feet long. If the person is at a distance of ten feet from the lamppost at that moment, what is the person's height?



31. With 100 feet of string out, a kite is 64 feet above ground level. When the girl flying the kite pulls in 40 feet of string, the angle formed by the string and the ground does not change. What is the height of the kite above the ground after the 40 feet of string have been taken in?



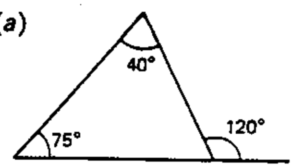
32. Find x. Explain your answers.



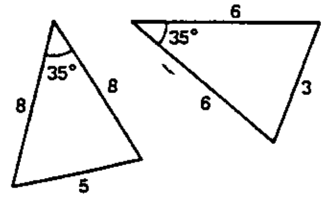
33. If you double the lengths of the legs of a right triangle, what happens to
(a) the length of the hypotenuse? Prove it.
(b) the area of the triangle? Prove it.

34. Without making another drawing, explain what is wrong in each picture.

(a)



(b)



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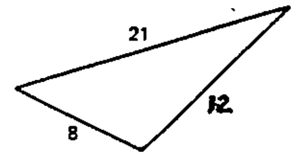

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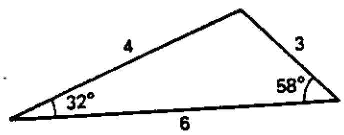

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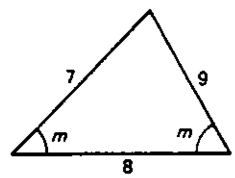

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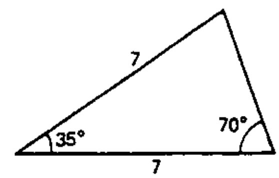
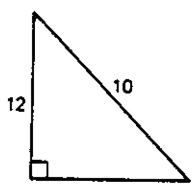
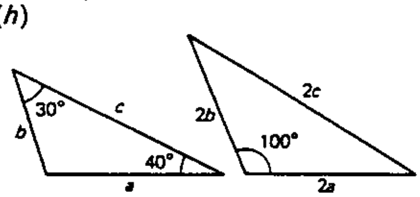


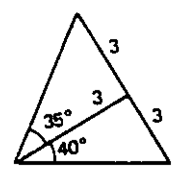
(g)



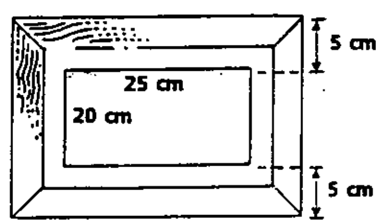
(h)



(i)



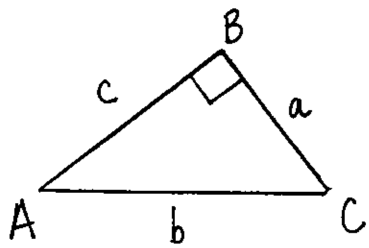
35. A picture measures 20 centimeters by 25 centimeters. The width of the matting and frame at top and bottom is 5 centimeters. How wide should the matting and frame be at the sides so that the framed picture is similar to the picture alone?



Math 20

Trigonometry Review #1

1. If the length of a leg of an isosceles right triangle is 5, determine the length of the hypotenuse of the triangle.

2. 

Express the ratio $\frac{b}{c}$ as a trigonometric function of an angle.

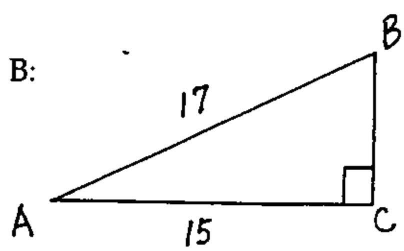
3. Find the exact value of $\sec\frac{\pi}{6}$

4. If $\cot A = \frac{3}{5}$, find $\csc A$

5. Express $\cos\frac{\pi}{7}$ as a function of the complementary angle.

6. If $\csc 71° = \sec B$, find B

7. Find sec B:



8. Find θ if $\sin\theta = \frac{1}{2}$ (Give θ in radians)

9. If $\csc A = \frac{7}{2}$, find $\tan A$

10. If $\cos 18° = \sin A$, find A

11. If $\sin B = \frac{5}{12}$, find $\cot B$

12. If $\tan \alpha = \sqrt{3}$, find α (Give α in degrees)

13. Find the value of θ in degrees (round to the nearest tenth of a degree) using your calculator:

(a) $\sin \theta = 0.8191$ (b) $\cos \theta = 0.0175$

14. Find the value of θ in radians (round to the nearest hundredth of a radian) using your calculator:

(a) $\sin \theta = 0.8746$ (b) $\cos \theta = 0.8746$

15. If $\tan A = \frac{1}{3}$, find:

(a) $\sin A$ (b) $\cos A$

16. Express $\sec 17°$ as a function of the angle complementary to the given angle.

17. If $\cot A = \frac{3}{5}$, find $\tan A$

18. Find the sine of the angle whose cosine is $\frac{1}{5}$

19. Find the tangent of the angle whose sine is x

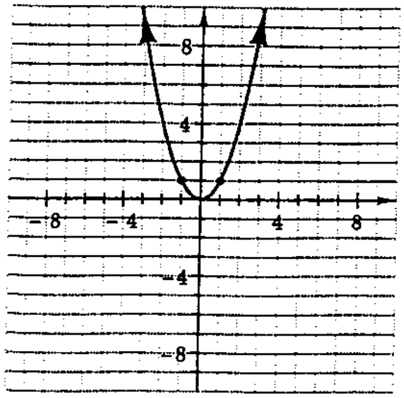
20. Find the **exact** value of each of the following (no calculators):

(a) $\sin 60°$ (b) $\cos \frac{\pi}{6}$

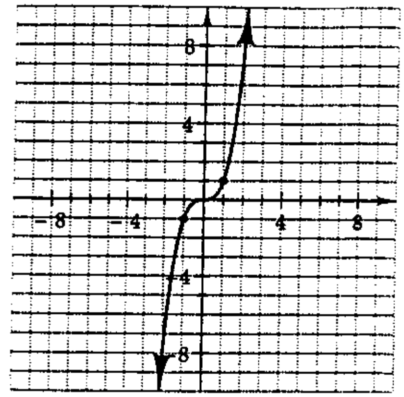
(c) $\tan 45°$ (d) $\cot \frac{\pi}{3}$

(e) $\sec 30°$ (f) $\csc \frac{\pi}{4}$

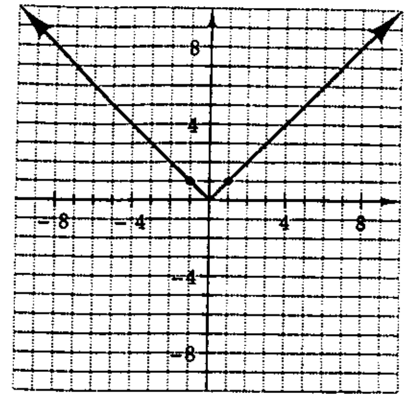
BASE GRAPHS



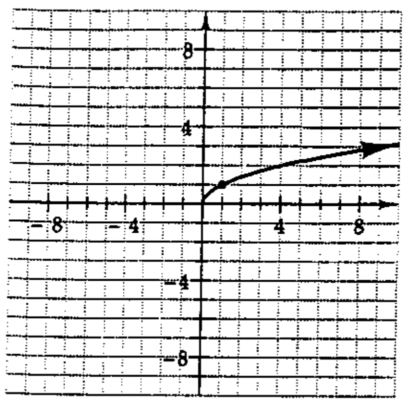
$y = x^2$



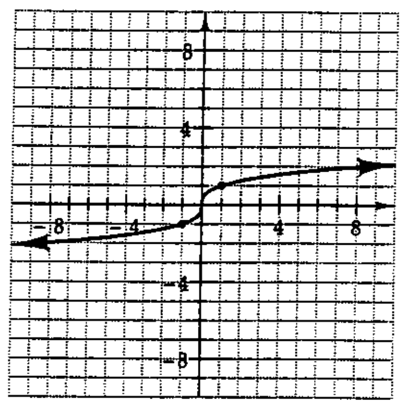
$y = x^3$



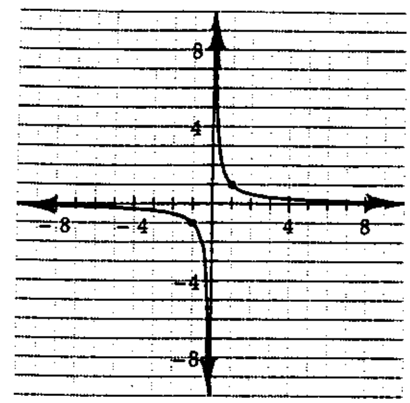
$y = |x|$



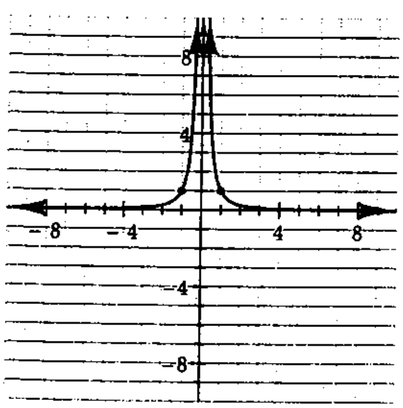
$y = \sqrt{x}$



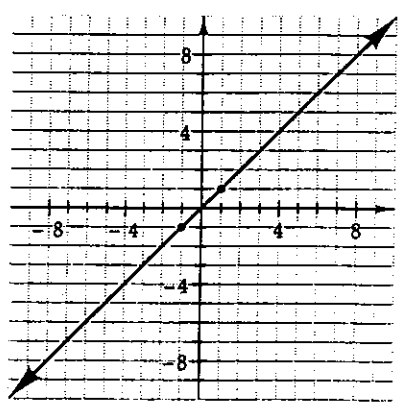
$y = \sqrt[3]{x}$



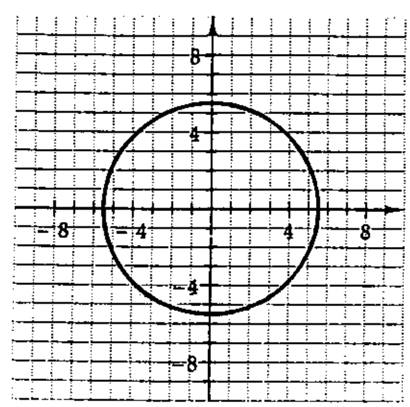
$y = \frac{1}{x}$



$y = \frac{1}{x^2}$



$y = x$



$x^2 + y^2 = r^2$

Math 20

Base Graphs & Translations
Homework Assignment

Graph each of the following functions by recognizing the base graph and transformations. Clearly indicate the key points and label your axes. Use graph paper. No graphing calculators!

1. $y = x^2 + 3$

2. $y = -3\,|x|$

3. $f(x) = \sqrt[3]{x+2} + 1$

4. $g(x) = \sqrt{x-4}$

5. $y = \frac{1}{2}(x+3)^3$

6. $y = \frac{3}{x}$

7. $f(x) = \frac{1}{(x-2)^2}$

8. $y = 2\sqrt{x} - 4$

9. $f(x) = -(x-6)^2$

10. $y = \sqrt[3]{x} - 2$

11. $y = 3 - |x|$

12. $(x-1)^2 + y^2 = 4$

13. $y = -\frac{4}{x^2}$

14. $y = -x^3 + 2$

15. $f(x) = \frac{1}{x+3} - 4$

16. $(x+1)^2 + (y-2)^2 = 1$

17. $y = \frac{1}{3}x - 5$

18. $g(x) = 2\,|x-1| - 3$

19. $y = 4 - x$

20. $y = \frac{3}{x-2}$

FUNCTIONS

Objectives:

1. Definition of a function
2. Vertical line test
3. Domain and range
4. Even and odd functions

Algebra often attempts to determine the relationship between two variables. One of the variables is considered the **independent variable** and the other is the **dependent variable.** For example the cost to mail a package depends on its weight, hence the weight is the independent variable and the cost is the dependent variable. The notation used for this example would be:

C(w) read "C of w" where w is the weight and C(w) is the cost.

Another example is your electric bill depends on the number of kilowatt hours used during the month. E(n) where n is number of kilowatt hours and E(n) is the electric bill. Do you know which is the dependent variable? If you answer the electric bill, you are correct.

A. DEFINITION:

A function is a process by which each value assigned to x, the independent variable, produces one and only one value for y, the dependent variable.

Example 1. Let y = 2x + 1. The function or process here is to double a number then add one. Arbitrarily let x = 3, then notice that twice three plus 1 is 7 (i.e. $2 \cdot 3 + 1 = 7$). Hence if x is 3 then the only value that is produced by the process for y is 7, so y is a function of x.

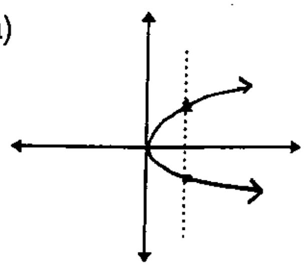
Example 2. Let $y = \pm\sqrt{x}$. Arbitrarily let x = 16, then $\pm\sqrt{16} = \pm 4$.
Hence if x is 16 then there are two possible values for y, both -4 and 4, so y is <u>not</u> a function of x.

B. VERTICAL LINE TEST for FUNCTIONS:

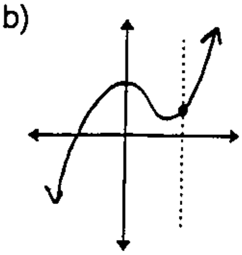
If a vertical line at any position intersects the graph in more than one point, the graph is **not** the graph of a function.

Example 3:

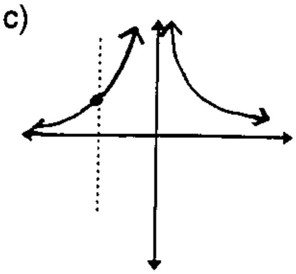
a)



b)



c)



In this case example 3a) is not a function because the vertical line (dotted) crosses the curve in two locations. Examples 3b) and 3c) are functions since the vertical line **only** crosses the curve in **one** location.

C. DOMAIN & RANGE:

The domain is the set of values for the independent variable and the range is the set of values generated by the function.

Example 4: Find the domain and range of $f(x) = \sqrt{x}$.

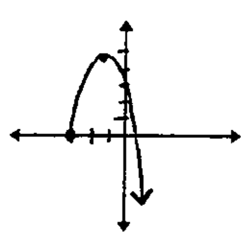
Since $f(x)$ contains a radical the values of x must be either positive or zero. Using interval notation, the domain is $[\,0, \infty)$.

The only values generated by the function are either zero or positive numbers. Using interval notation, the range is $[0, \infty)$.

Example 5: Find the domain of $f(x) = \dfrac{1}{x-5}$.

Since $f(x)$ contains a fraction any value that yields zero in the dominator must be excluded. The domain is the set of all real numbers except 5.

Example 6: State the domain and range of the graph, use interval notation.



The values for x begin at -3, hence the domain is $[-3, \infty)$.

The values for y begin at negative infinity and end at 5, hence the range is $(-\infty, 5\,]$.

E. EVEN & ODD FUNCTIONS:

A function is said to be **even** if its graph is symmetric with respect to the y-axis and to be **odd** if its graph is symmetric with respect to the origin.

Example 7: Name as many as possible of your base graphs that are even functions.

Possible answers are $f(x) = x^2$, $f(x) = |x|$, *and* $f(x) = \frac{1}{x^2}$.

Algebraic test for even and odd functions:
A function *f(x)* is even if, for each x in the domain of *f*, *f(-x) = f(x)* .
A function *f(x)* is odd if, for each x in the domain of *f*, *f(-x) = - f(x)* .

Example 8: The function $f(x) = 2x^3$ is odd because replacing all the x by -x yields $f(-x) = 2(-x)^3 = -2x^3 = -f(x)$.

Example 9: The function $f(x) = \frac{1}{x^2 - 4}$ is even because replacing all the x by -x yields $f(-x) = \frac{1}{(-x)^2 - 4} = \frac{1}{x^2 - 4} = f(x)$.

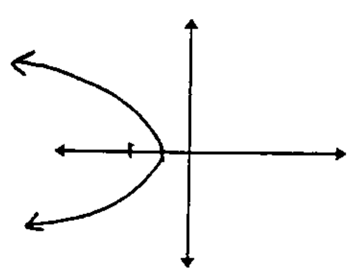
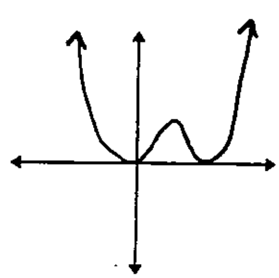
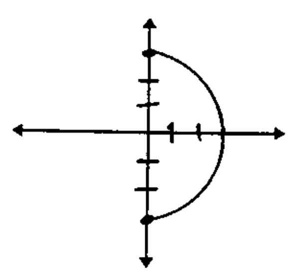
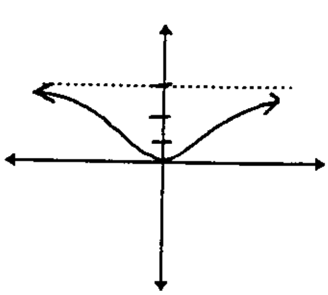
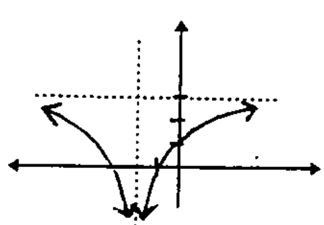
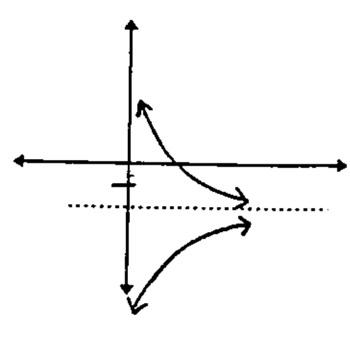
FUNCTION HANDOUT -- Homework assignment

For each of the following, determine if y is a function of x. If it is not, give **two specific points** which prove it is not a function.

1. $y = \frac{1}{2}x^2$
2. $2x = y^2$
3. $y = |x|$
4. $x = |y|$
5. $y = x^3$
6. $x = y^3$
7. $y = 1 - \frac{1}{4}x$
8. $x = \frac{4 - y}{4}$
9. $x = 2$
10. $y = -3$
11. $y = \sqrt{9 - x^2}$
12. $y = \frac{5}{x - 6}$
13. $y = -\sqrt{x + 3}$
14. $y = \frac{1}{(x+2)^2}$
15. $y = -\frac{1}{x} + 2$

16. Find the domain and range for problems 1,3,5,7,9,10,13,14, and 15. Use interval notation.

17. Which of the following graphs represent functions? Name the domain and range for each using interval notation.

QUADRATIC FUNCTIONS

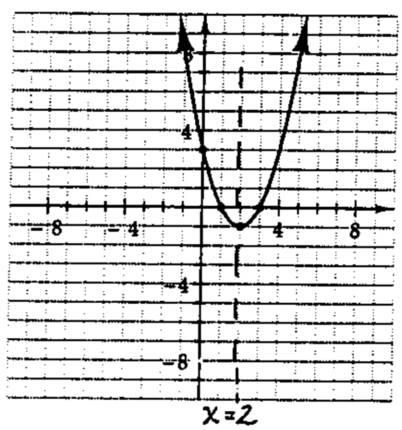
Objectives:

1) definition of a quadratic function
2) find the x and y intercepts
3) find the vertex using complete the square
4) naming the axis of symmetry

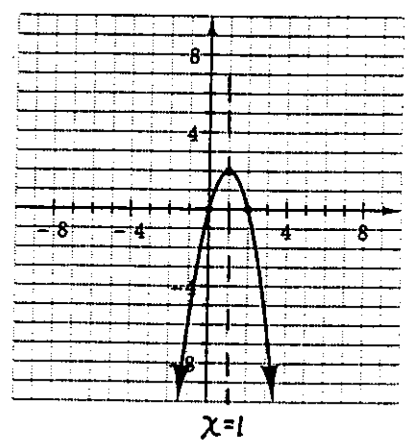
DEFINITION -- A function of the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers with $a \neq 0$, is called a **quadratic function**. The graph of such a function is called a **parabola**.

The important properties of a quadratic function are its intercepts, vertex, and axis of symmetry. See the two examples below. The intercepts, vertex and axis of symmetry are marked on each graph.

$f(x) = x^2 - 4x + 3$



$f(x) = 4x - 2x^2$



A. THE INTERCEPTS:

To find the y-intercept, set x = 0, then solve for y.
To find the x-intercept, set $f(x) = 0$, then solve for x.

Example 1: Let $f(x) = x^2 - 4x + 3$, clearly state in coordinate form the intercepts. Also name the equation of the axis of symmetry.

To find the y-intercept, set x = 0.

$$y = 0^2 - 4(0) + 3 = 3,$$

hence the **y-intercept is (0, 3).**

To find the x-intercept, set *f(x)* = 0.

$$0 = x^2 - 4x + 3$$
$$0 = (x - 3)(x - 1)$$
$$x = 1,\ 3$$

Hence the **x-intercepts are (1, 0) and (3, 0).**

The axis (or line) of symmetry is such that if the curve were folded over on this line, its left half would coincide with its right half. So the line of symmetry is halfway between the two x-intercepts, that is, the line **x = 2 is the axis of symmetry**.

B. THE VERTEX:

To find the coordinates of the vertex, it is necessary for the function to be in the form $f(x) = a(x - h)^2 + k$ where the **vertex is *(h, k)*** and "*a*" determines the shape and direction of the parabola. In order to put the quadratic function in this form it is necessary to complete the square.

Example 2: Let $f(x) = x^2 - 4x + 3$, find the coordinates of the vertex then use the intercepts and the vertex to sketch the graph..

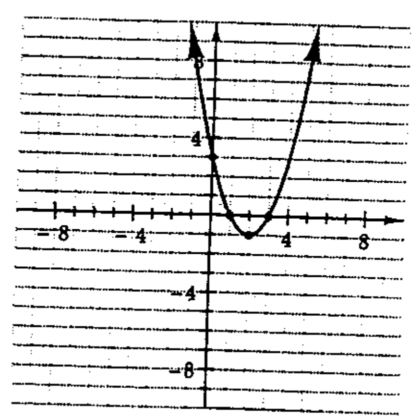
$$f(x) = \underbrace{x^2 - 4x + \quad}\ + 3 - \qquad \text{Complete the square by}$$

adding then subtracting 4

$$f(x) = \underbrace{x^2 - 4x + 4}\ + 3 - 4 \qquad \text{Factor the trinomial}$$

$$f(x) = (x - 2)^2 - 1$$

Hence the **vertex is (2, -1).**



Example 3: Determine the vertex, x and y intercepts, and axis of symmetry for the graph of $f(x) = -x^2 + 3x + 10$.

Setting x = 0, leads to y = 10, hence the **y-intercept is (0, 10).**

Setting *f(x)* = 0, leads to

$$0 = -x^2 + 3x + 10$$
$$0 = -\left(x^2 - 3x - 10\right)$$
$$0 = -(x-5)(x+2)$$
$$x = 5, \ -2$$

hence the **x-intercepts are (5, 0) and (-2, 0).**

To find the vertex, complete the square:

$$f(x) = -\left(x^2 - 3x\right) + 10$$
$$f(x) = -\left(x^2 - 3x + \frac{9}{4}\right) + 10 + \frac{9}{4}$$
$$f(x) = -\left(x - \frac{3}{2}\right)^2 + \frac{49}{4},$$

hence the vertex is $\left(\frac{3}{2}, \frac{49}{4}\right)$.

The axis of symmetry goes through the vertex, so the equation is $x = \frac{3}{2}$.
Notice that "a" is -1 so the direction of the parabola is downward. **The vertex of a parabola is also called its maximum or minimum point.**

C. SHORT CUT FORM TO FIND THE VERTEX:

If the quadratic is in the form $f(x) = ax^2 + bx$, where the constant term is missing, or if the x-intercepts are easy to find then the easiest way to find the x coordinate of the vertex is to find the x-intercepts. The vertex is always located at their midpoint. Substitute the midpoint into the function will then yield the y coordinate of the vertex.

Example 4: Find the vertex and graph $f(x) = x^2 + 8x$.

By factoring the x-intercepts are 0 and -8. Their midpoint is -4. Therefore the x coordinate of the vertex is -4. Now substitute -4 into the function.

$$y = (-4)^2 + 8(-4) = 16 - 32 = -16.$$

The vertex is (- 4, - 16)

QUADRATIC FUNCTIONS - homework assignment

In exercise 1-8 graph the quadratic function defined by the equation. On the graph indicate
a) the coordinates of the x and y intercepts
b) the equation of the axis of symmetry
c) the coordinates of the vertex
d) the range of the function

1. $f(x) = x^2 + 2x - 24$
2. $f(x) = -x^2 - 4x + 5$
3. $f(x) = 2x^2 + 8x$
4. $f(x) = x^2 - 4$
5. $f(x) = 4x^2 - 4x - 3$
6. $f(x) = x^2 - 5x$
7. $f(x) = x^2 + 10x + 25$
8. $f(x) = -x^2 - 4$

In exercise 9-12 determine the vertex of each function by completing the square.

9. $f(x) = x^2 - 2x + 3$
10. $f(x) = x^2 + 6x - 1$
11. $f(x) = -x^2 + x + 2$
12. $f(x) = 2x^2 + 4x - 5$

13. What is the domain for all the functions in exercises 1-10?

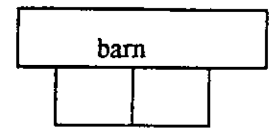
14. Find two positive numbers whose sum is 20 and whose product is a maximum.

15. The height of a ball thrown directly up from a roof 144 ft high with an initial velocity of 128 ft/sec is given by the formula $y = 144 + 128t - 16t^2$. What is the maximum height attained by the ball?

16. The sum of the base and altitude of a triangle is 12 inches. Find the base and altitude if the area is to be a maximum.

17. Find the maximum possible area of a rectangle with a perimeter of 100 feet.

18. A farmer wants to make a rectangular enclosure along the side of a barn and then divide the enclosure into two pens with a fence constructed at right angle to the barn. If 300 yards of fencing are available, what are the dimensions of the largest section that can be enclosed?



19. A bus tour charges a fare of $10 per person and carries 200 people per day. The manager estimates that she will lose 10 passengers for each $1 increase in fare. Find the most profitable fare for her to charge.

20. A caterer will charge a certain club $20 per member to cater a dinner if 150 members attend. However, the cost per member will be reduced by 80 cents for each member attending in excess of 150. What is the maximum revenue that the caterer can make? If you were the caterer, what upper limit would you set for the number of members that can attend the dinner?

FACTORING

Objective:

1. Factor greatest common factor
2. Factor trinomials
3. Factor by grouping
4. Factor difference of perfect squares
5. Factor sum and difference of cubes
6. Factor with fraction and negative exponents
7. Factor with variable exponents

The first five types listed above are treated as review. It is assumed that you have been exposed to many of these types and are accurate at factoring them. The intention here is to combine them into multi-step problems and include trigonometry. The last two types are treated as new topics.

Review examples:

Factor each completely.

a) $y^4 - 2y^3 - 9y^2 + 18y$ — 1) factor the common "y"

$= y\left(y^3 - 2y^2 - 9y + 18\right)$ — 2) factor by grouping

$= y\left[y^2(y-2) - 9(y-2)\right]$ — 3) factor the common *(y - 2)*

$= y(y-2)\left(y^2 - 9\right)$ — 4) factor the difference of squares

$= y(y-2)(y-3)(y+3)$

b) $\sin^3 x + 27$ — 1) factor the sum of cubes

$= (\sin x + 3)\left(\sin^2 x - 3\sin x + 9\right)$

c) $18\cos^2 x \tan x - 57\cos x \tan x + 24\tan x$ — 1) factor common "3tan x"

$= 3\tan x\left(6\cos^2 x - 19\cos x + 8\right)$ — 2) factor the trinomial

$= 3\tan x(2\cos x - 1)(3\cos x - 8)$

FACTORING WITH FRACTIONAL OR NEGATIVE EXPONENTS:

Rule: Identify any base(s) common to all the terms. The greatest common factor is the factor with the smallest exponent.

Example 1: For each of the following identify the common factor(s) and the GCF.

a) $x^{-3} + 2x^{-1}$ — The common factor is "x" and the GCF is x^{-3}.

b) $x^{-2}y^{2} - x^{2}y^{-1}$ — The common factors are "x" and "y" and the GCF is $x^{-2}y^{-1}$.

c) $4x^{-1}y^{-1/2} + 2xy^{1/2}$ — The common factors are "2", "x" and "y" and the GCF is $2x^{-1}y^{-1/2}$.

Example 2: Factor each completely:

a) $5x^{-1} + 3x$ — The GCF is x^{-1}

hence $5x^{-1} + 3x$

$= x^{-1}\left(5 + 3x^{2}\right)$

b) $x^{1/2}(x+2)^{-3} + 3x^{-1/2}(x+2)^{-2}$ — The GCF is $x^{-1/2}(x+2)^{-3}$

hence $x^{1/2}(x+2)^{-3} + 3x^{-1/2}(x+2)^{-2}$

$= x^{-1/2}(x+2)^{-3}\left[x + 3(x+2)\right]$

$= x^{-1/2}(x+2)^{-3}\left[4x + 6\right]$

$= 2x^{-1/2}(x+2)^{-3}\left[2x + 3\right]$

FACTORING - Homework Assignment

Factor each completely.

1. $4x^3 - 8x^2 - 9x + 18$

2. $(2x+5)^2 + 9(2x+5)$

3. $25x^2 - 40xy + 16y^2$

4. $4\cos^2 x - 13\cos x + 3$

5. $(x-y)^2 - 4$

6. $(\tan x - 1)^2 + 3(\tan x - 1)$

7. $\sin^2 x + 2\sin x\cos x + \cos^2 x - 9$

8. $\tan^3 x + 1$

9. $8x^3 - 125$

10. $(x-y)^3 + 27$

11. $x^{2/3} - 5x^{1/3} - 24$

12. $x - 5x^{1/2} + 6$

13. $(x+3)^{-2} - 3(x+3)^{-1} - 10$

14. $x^2 - a^2 + x - a$

15. $(x+2)^{-1}(x-1)^{-3} - (x+2)^{-2}(x-1)^{-2}$

16. $2x^{-3/2} - 8x^{1/2}$

17. $(y+2)^{1/5} - (y+2)^{-4/5}$

18. $x^{1/2}(x-3)^{-1/2} + x^{-1/2}(x-3)^{-3/2}$

19. $x^{n+1} + 3x^n$

20. $x^{2n} + 6x^n + 9$

21. $y^{4n} - 81$

22. $x^{2n+2} + x^{n+2} - 30x^2$

23. $6x^{n+2} - 3x^{n+1} - x^n$

RATIONAL EXPRESSIONS

Simplify each. Express all answers in reduced form.

1. $\dfrac{1-x}{x^2-1}$

2. $\dfrac{3-\tan x}{\tan^2 x - 9}$

3. $\dfrac{8x^3-1}{2x-1}$

4. $\dfrac{64-\cos^2 x}{2\cos x - 16}$

5. $\dfrac{\left(x^2+1\right)-\left(a^2+1\right)}{x-a}$

6. $\dfrac{(x+h)^2-25}{(x+h)-5}$

7. $\dfrac{\sin^3 x - 27}{\sin^3 x - 3\sin^2 x + 3\sin x - 9}$

8. $\dfrac{2(2+h)-6(2+h)}{2h(2+h)}$

9. $\dfrac{(x+1)^2-2x(x+1)}{(x+1)^3}$

10. $\dfrac{\tan^2 x + 4\tan x - 5}{5 - 4\tan x - \tan^2 x}$

11. $\dfrac{(y-1)^2}{(y+2)^2} \cdot \dfrac{y^2-4}{y^2-1}$

12. $\dfrac{5}{\sin^2 x} \div \dfrac{5}{\sin x}$

13. $\dfrac{2x^2-2ax+2a^2}{(ax)^3} \div \dfrac{a^3+x^3}{ax^3}$

14. $\dfrac{\sin^2 x + 2\sin x}{6\tan x} \cdot \dfrac{\tan^2 x}{\sin^2 x - 4}$

15. $\dfrac{y^3-x^3}{(y-x)^3} \div \dfrac{x^2+xy+y^2}{x^2-2xy+y^2}$

16. $\dfrac{-4}{3-\sin x} \div \dfrac{12}{\sin x - 3}$

17. $\dfrac{\tan x - 1}{2-2\tan x} \div \left(\dfrac{\tan x - 1}{3-3\tan x} \div \dfrac{\tan x - 1}{4-4\tan x}\right)$

18. $\dfrac{n+1}{3^{n+1}} \div \dfrac{n}{3^n}$

19. $\left(\dfrac{n+2}{n+1} \cdot \dfrac{1}{4^n}\right) \div \left(\dfrac{n+1}{n} \cdot \dfrac{1}{4^{n+1}}\right)$

20. $\dfrac{n+1}{3^{n-1}} \cdot \dfrac{3^{n+1}}{n^2+n}$

21. $\dfrac{2x+3}{5x} - \dfrac{2x-1}{10x} + \dfrac{4}{x}$

22. $\dfrac{2x-5}{4-3x} + \dfrac{2-x}{3x-4}$

23. $\left(4+\dfrac{1}{a-1}\right) \div \left(\dfrac{2}{a-1}+3\right)$

24. $\dfrac{2a}{a^2-1} - \dfrac{a+1}{a-1} + 1$

25. $1 - \left(\dfrac{a^x - a^{-x}}{a^x + a^{-x}}\right)$

26. $\left(\dfrac{a^x + a^{-x}}{2}\right)^2 - \left(\dfrac{a^x - a^{-x}}{2}\right)^2$

27. $\dfrac{1}{x^n} + \dfrac{2}{x^{n+1}}$

28. $\dfrac{a}{x^{n+1}} + \dfrac{b}{x^n}$

29. $\dfrac{2}{x^{n-1}} - \dfrac{1}{x^{n+1}}$

30. $\dfrac{\tan x}{2} - \dfrac{\tan x - 1}{2}$

31. $\dfrac{1}{\tan x - 1} + \dfrac{2}{1 - \tan x}$

32. $\dfrac{2-\tan x}{3-2\tan x} - \dfrac{4}{2\tan x - 3}$

33. $\dfrac{5}{2\tan^2 x} - \dfrac{3}{4\tan x}$

34. $\dfrac{1}{\tan x} - \tan x$

35. $\dfrac{4\cos x}{5} + \dfrac{2}{5\cos x} - \dfrac{\cos x}{2}$

36. $\dfrac{\tan x}{\tan^2 x - 4} - \dfrac{1}{\tan x + 2}$

37. $\dfrac{\cos x}{\cos x - 1} - \dfrac{2}{\cos^2 x - 1}$

38. $\dfrac{\cos x + 1}{2\cos x + 6} + \dfrac{4 - \cos^2 x}{2\cos^2 x + 2\cos x - 12}$

39. $\dfrac{1}{h}\left(\dfrac{1}{x+h} - \dfrac{1}{x}\right)$

40. $\dfrac{1}{h}\left[\dfrac{2}{(x+h)^2} - \dfrac{2}{x^2}\right]$

Algebra Review

I. Complex Fractions

Recall that a complex fraction is one in which there are fractions in the numerator, denominator, or both. The fastest way to simplify a complex fraction is to multiply the numerator and denominator by the least common denominator of all of the fractions which appear inside.

Example 1

To simplify the complex fraction $\dfrac{3 - \dfrac{1}{x^2}}{2 - \dfrac{3}{x}}$ we would multiply the numerator and denominator by the LCD of the fractions which is x^2:

$$\frac{\left(3 - \dfrac{1}{x^2}\right)x^2}{\left(2 - \dfrac{3}{x}\right)x^2} = \frac{3x^2 - 1}{2x^2 - 3x}$$

Example 2

To simplify the complex fraction $\dfrac{\dfrac{b}{b-a} - \dfrac{a}{b+a}}{\dfrac{b^2+a^2}{b^2-a^2}}$ we would multiply the numerator and denominator by the LCD of the fractions which is $b^2 - a^2 = (b-a)(b+a)$:

$$\frac{\left(\dfrac{b}{b-a} - \dfrac{a}{b+a}\right)(b-a)(b+a)}{\left(\dfrac{b^2+a^2}{b^2-a^2}\right)(b-a)(b+a)} = \frac{b(b+a) - a(b-a)}{b^2 + a^2}$$

$$= \frac{b^2 + ab - ab + a^2}{b^2 + a^2} = \frac{b^2 + a^2}{b^2 + a^2} = 1$$

II. Negative and Fractional Exponents

To simplify expressions with negative or fractional exponents, we use the same properties of exponents with which you are already familiar as well as the following property: $a^{-n} = \frac{1}{a^n}$. Expressions with fractional exponents can also be written in radical form but we will be studying that later in the course. For now, we will simplify by using the properties of exponents.

Example 1

$$2^{1/2} \cdot 2^{3/2} = 2^{1/2 + 3/2} = 2^{4/2} = 2^2 = 4$$

Example 2

$$\frac{5^{1/3}}{5^{4/3}} = 5^{1/3 - 4/3} = 5^{-3/3} = 5^{-1} = \frac{1}{5}$$

Example 3

$$\left(a^{-2}b^{6}\right)^{-1/2} = \left(a^{-2}\right)^{-1/2}\left(b^{6}\right)^{-1/2} = a^{1}\, b^{-3} = a \cdot \frac{1}{b^3} = \frac{a}{b^3}$$

Example 4

$$(x+y)^{5/3}(x+y)^{-2/3} = (x+y)^{5/3 + -2/3} = (x+y)^{3/3} = x+y$$

Example 5

$$\left(\frac{b^{1/3}}{b^{4/3} + b^{7/3}}\right)^{-1} = \frac{1}{\frac{b^{1/3}}{b^{4/3} + b^{7/3}}} = \frac{b^{4/3} + b^{7/3}}{b^{1/3}} = \frac{b^{4/3}}{b^{1/3}} + \frac{b^{7/3}}{b^{1/3}}$$

$$= b^{4/3 - 1/3} + b^{7/3 - 1/3} = b^{3/3} + b^{6/3} = b + b^2$$

ALGEBRA REVIEW
HOMEWORK ASSIGNMENT

I. COMPLEX FRACTIONS

Simplify each of the following complex fractions:

1. $$\frac{z + 4 - \frac{5}{z}}{z + 1 - \frac{2}{z}}$$

2. $$\frac{1 + \frac{3}{x + 1}}{\frac{4}{x^2 - 1}}$$

3. $$\frac{\frac{1}{\sin\theta + 1} + \frac{1}{\sin\theta - 1}}{\frac{1}{\sin\theta - 1} - \frac{1}{\sin\theta + 1}}$$

4. $$\frac{\frac{3}{(x + h)^2} - \frac{3}{x^2}}{h}$$

5. $$\frac{\frac{x}{1 + x} - \frac{1 - x}{x}}{\frac{x - 1}{x} - \frac{x}{x + 1}}$$

6. $$\frac{1 - \frac{\sin x}{\cos x}}{\cos x - \frac{\sin^2 x}{\cos x}}$$

7. $$\frac{1 + \frac{1}{n}}{\frac{1}{n+1} - 1}$$

8. $$\frac{\frac{2^{n+1}}{n\,3^n}}{\frac{2^n}{(n - 1)\,3^{n-1}}}$$

II. FRACTIONAL AND NEGATIVE EXPONENTS

Simplify each of the following expressions. Write your final answer without negative exponents.

9. $\dfrac{xy^{-1} + x^{-1}y}{x^{-1}y - xy^{-1}}$

10. $\left(x^{-1} + y^{-1}\right)^{-1}$

11. $\dfrac{x^{-1/4}}{x^{1/2}}$

12. $\dfrac{y + 3}{(y + 3)^{-1/2}}$

13. $\left(\dfrac{x^{1/4}y^{-2/3}z^{0}}{x^{3/4}y^{-2/3}z^{1/2}}\right)^{-3/2}$

14. $\left(x^{1/2} - x^{-1/2}\right)^{2}$

15. $\dfrac{x^{1/3} + x^{-2/3}}{x^{1/3} - x^{-2/3}}$

16. $\left(a^{x^2 + 2x} \cdot a\right)^{\frac{1}{x+1}}$

DIFFERENCE QUOTIENT HANDOUT

Write the difference quotient for each function in simplest form.

1. $f(x) = 3 - 5x$

2. $f(x) = \dfrac{2x-3}{4}$

3. $f(x) = 5x^2$

4. $f(x) = x + x^2$

5. $f(x) = 5 - 2x^2$

6. $f(x) = 3(x-1)^2$

7. $f(x) = \dfrac{1}{x}$

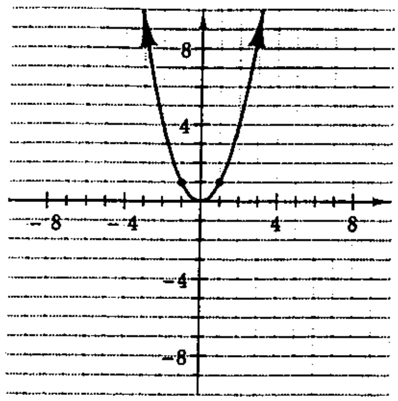
8. $f(x) = \dfrac{2}{x+3}$

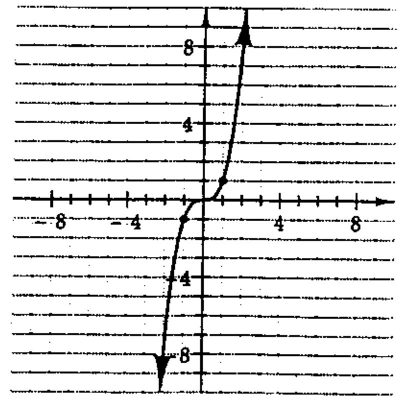
9. $f(x) = \dfrac{1}{x^2}$

10. $f(x) = 10$

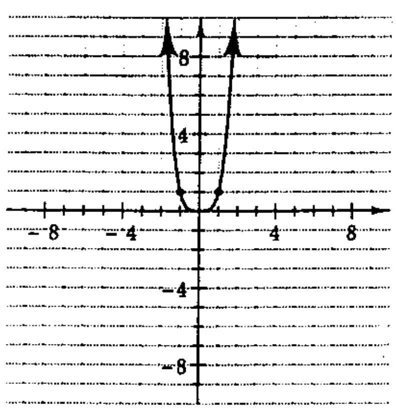
Graphs of Polynomials

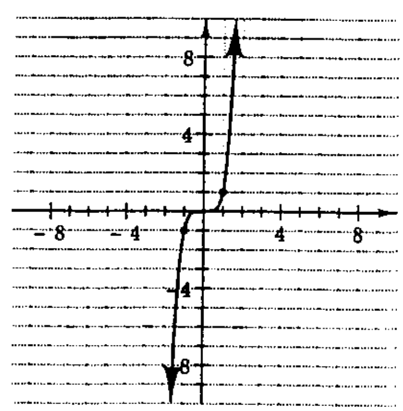
From your experience with base graphs, you are already familiar with the graph of the polynomials $y = x^2$ and $y = x^3$:





Now we will consider the graphs of polynomials of the form $y = x^n$ where n is a positive integer larger than 3 ($n = 4, 5, 6, \ldots$). For these larger values of n, the graph of $y = x^n$ is similar to the graph of $y = x^2$ if n is even, and it is similar to the graph of $y = x^3$ if n is odd. Note that when n is even, the graph of $y = x^n$ is symmetric with respect to the y-axis and passes through the points $(-1,1)$, $(0,0)$, and $(1,1)$ as does $y = x^2$. When n is odd, the graph is symmetric with respect to the origin and passes through the points $(-1,-1)$, $(0,0)$, and $(1,1)$ as does $y = x^3$. In both cases as the exponent n increases, the graph becomes flatter on the interval $[-1,1]$ and rises or falls more quickly for x-values less than -1 or greater than 1. Below are the graphs of $y = x^4$ and $y = x^5$:





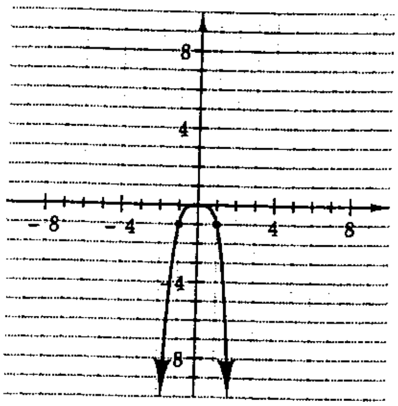
To graph certain variations of $y = x^n$ we may use the graphing techniques previously discussed with the base graphs. This includes stretching and shrinking vertically, shifting horizontally and vertically, and reflecting about the x-axis.

Example 1

Use the graph of $y = x^4$ to graph each of the following:

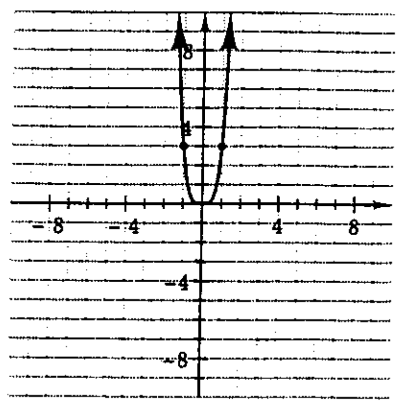
(a) $y = -x^4$

We reflect the graph $y = x^4$ about the x-axis to obtain the graph in part (a).



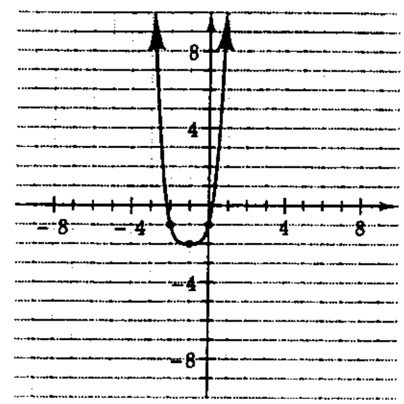
(b) $y = 3x^4$

The coefficient of 3 is a vertical stretch; thus key points for this graph are (–1,3), (0,0), and (1,3).



(c) $y = (x + 1)^4 - 2$

This graph is shifted one unit to the left and down two units from the base graph $y = x^4$.



Note that the only real-number zero of $y = x^n$ is 0; and that when n is odd, the graph crosses the x-axis at x = 0, while the graph turns around and stays on the same side of the x-axis when n is even (we say that the graph "bounces" at this zero). This behavior occurs because x-values change sign as x passes through 0, so y changes sign when n is odd and y keeps the same sign when n is even. This type of analysis is applicable to all x-intercepts and indicates that the multiplicity of each real-number zero reveals whether or not the graph crosses the x-axis at such intercepts, according to the following theorem:

Graph of y = P(x) Near X-Intercepts	If b is a real number zero with multiplicity n of y = P(x), then the graph of y = P(x) crosses the x-axis at x = b if n is odd, while the graph turns around and stays on the same side of the x-axis at x = b is n is even.

One other theorem is needed to complete the graphs of polynomials:

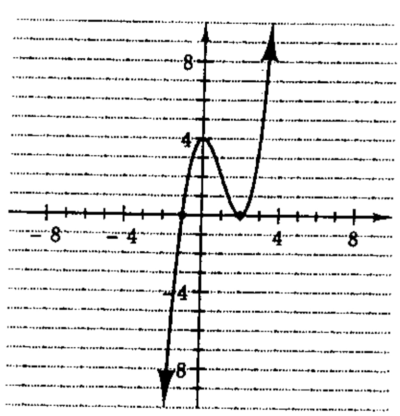
Behavior for Large $\lvert x \rvert$	The graph of the n-th degree polynomial $y = a_n x^n + \ldots + a_1 x + a_0$ resembles the graph of $y = a_n x^n$ when $\lvert x \rvert$ is large.

This behavior occurs because for x-values far from the origin ("end behavior"), the leading term $a_n x^n$ is much larger than the sum of all of the other terms in the polynomial. We say that this term "dominates" the polynomial.

To use these theorems, we need to express $P(x)$ in factored form. Then the x-intercepts (or real-number zeros) can be obtained from the Factor Theorem, while the behavior of the graph at an x-intercept can be determined from the multiplicity of that zero. The behavior of the graph when $\lvert x \rvert$ is large is obtained using the above theorem and knowledge of the base graphs. Otherwise, polynomials are smooth, unbroken curves with "hills and valleys".

Example 2

To graph $y = (x+1)(x-2)^2$, we note that the x-intercepts are -1 and 2. Since -1 is a zero of multiplicity one, the graph crosses the x-axis at $(-1, 0)$. Since 2 is a zero of multiplicity two, the graph "bounces" at $(2,0)$. Since this is a third degree polynomial, the graph at the edges resembles that of $y = x^3$. The graph is illustrated below. Note that the only exact points that are graphed are the x-intercepts.



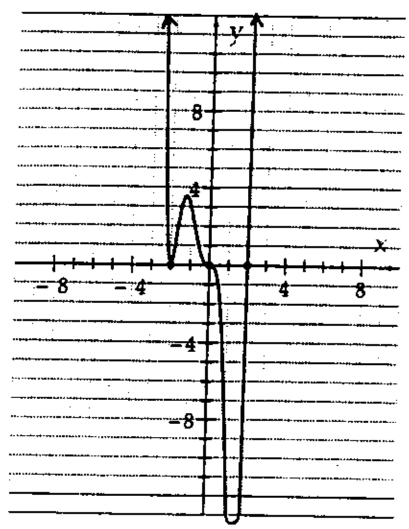
Example 3

To graph $y = x^6 + 2x^5 - 4x^4 - 8x^3$, we first factor:

$$x^6 + 2x^5 - 4x^4 - 8x^3 = x^3 (x^3 + 2x^2 - 4x - 8) = x^3 \left[x^2 (x+2) - 4(x+2) \right]$$

$$= x^3 \left[(x+2)(x^2 - 4) \right] = x^3 (x+2)^2 (x-2)$$

The x-intercepts are 0 (of multiplicity 3), – 2 (of multiplicity 2), and 2 (of multiplicity 1). Thus the graph crosses the x-axis at 0 and 2, and "bounces" at – 2. Since the degree of the polynomial is 6, the "end behavior" resembles that of $y = x^2$. The graph is illustrated below.



POLYNOMIAL GRAPHS
HOMEWORK ASSIGNMENT

Graph each of the following polynomials, using the base graphs and transformations as a guide.

1. $y = \frac{1}{2}x^3$

2. $y = 3(x+2)^4$

3. $y = 2 - (x-1)^5$

4. $y = (x-3)^6 + 4$

Graph each of the following polynomials based on information obtained from the x-intercepts.

5. $y = (x-2)^2(x+1)^2$

6. $y = x^4 - 5x^2 + 4$

7. $y = x^3 - 2x^2 - 3x$

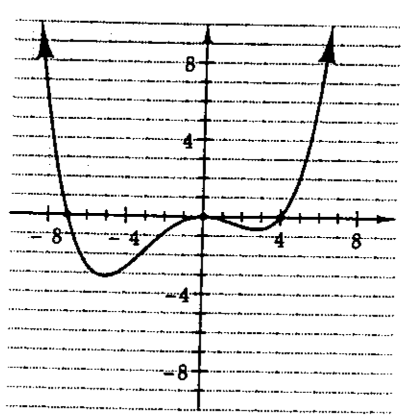
8. $y = x^3 - 2x^2 + 3x - 6$

9. If -1 is a zero of the polynomial $y = x^3 - x^2 - 10x - 8$, find all the x-intercepts and then graph the polynomial.

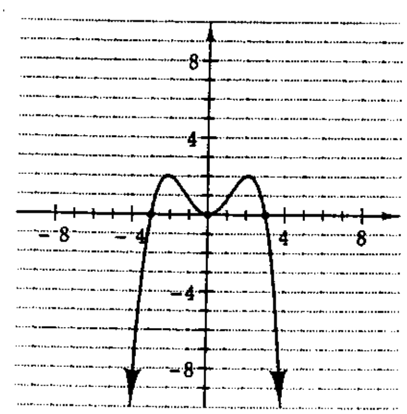
10. If 3 is a zero of multiplicity two of the polynomial $y = x^4 - 6x^3 + 7x^2 + 12x - 18$, find all the x-intercepts and then graph the polynomial.

Find the equation for each of the polynomials graphed below:

11.



12.



IDENTITES HANDOUT

Verify each of the following.

1. $\dfrac{1}{\tan x} + \dfrac{1}{\sin x} = \dfrac{\cos x + 1}{\sin x}$

2. $\dfrac{\sin x + \cos x}{\sin x} - \dfrac{\cos x + \sin x}{\cos x} = \dfrac{\cos^2 x - \sin^2 x}{\sin x \cos x}$

3. $\dfrac{\cos x + \cos x \sin x}{\cos x - \cos x \sin^2 x} = \dfrac{1}{1 - \sin x}$

4. $(\sec x - \tan x)^2 = \dfrac{1 - \sin x}{1 + \sin x}$

5. $\dfrac{1 - \sin^2 x}{\sin^2 x} = \csc^2 x - 1$

6. $\dfrac{4\cos^2 x - 1}{2\sin x \cos x - \sin x} = 2\cot x + \csc x$

7. $\dfrac{1}{\tan x - \sec x} + \dfrac{1}{\tan x + \sec x} = -2\tan x$

8. $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$

9. $\dfrac{\cot x}{1 + \cos x} + \dfrac{\cos x}{\sin x - \sin x \cos x} = 2\cos x \ \csc^3 x$

10. $\sin^3 x = \sin x - \cos^2 x \sin x$

11. $\dfrac{\tan x}{\sin x \cos x} - \dfrac{\sec^2 x}{\csc^2 x} = 1$

12. $\cos\left(2\pi - x\right) \tan\left(\pi + x\right) = \sin x$

13. $\dfrac{\cot x - \tan x}{\cot x + \tan x} = \cos 2x$

14. $\sin x = \dfrac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

15. $\sin(x + y) \cos y - \cos(x + y) \sin y = \sin x$

16. $2 \cos^3 x - \cos x = \dfrac{\cos^2 x - \sin^2 x}{\sec x}$

17. $\cot^2 y = \dfrac{1 + \cos 2y}{1 - \cos 2y}$

18. $\dfrac{1 - \cos x}{1 + \cos x} = (\cot x - \csc x)^2$

19. $2 \cot 2x = \cot x - \tan x$

20. $\dfrac{1 - \sin x}{1 + \sin x} = \sec^2 x - 2 \sec x \tan x + \tan^2 x$

21. $\dfrac{\sin x}{1 - \cot x} + \dfrac{\cos x}{1 - \tan x} = \sin x + \cos x$

RADICALS

Definition: A **radical** is said to be in **simplest form** if the following conditions are satisfied:

1. The radicand has no factor with an exponent greater than or equal to the root index (note that this may require you to prime-factor the coefficient).
2. The radicand is written with a positive coefficient.
3. The radicand does not contain a fraction.
4. The root index is reduced to the smallest possible value.
5. The denominator of a fraction does not contain a radical expression.

To simplify radicals, we use the following techniques:

I. Removing Factors

Use the property that $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$ and the property that

$$\sqrt[n]{a^n} = \begin{cases} a & \text{if } a \geq 0 \text{ or } n \text{ is odd} \\ |a| & \text{if } n \text{ is even} \end{cases}$$

It is also useful to prime-factor the numerical coefficient. If the numerical coefficient is negative (and n is odd), remove the factor of -1.

Example 1

(a) $\sqrt{25x^5y^4} = \sqrt{25x^4y^4}\,\sqrt{x} = 5x^2y^2\sqrt{x}$

(b) $\sqrt[3]{-80x^4y^5z^6} = \sqrt[3]{-1\cdot 2^4\cdot 5\ x^4y^5z^6} = \sqrt[3]{-1\cdot 2^3\,x^3y^3z^6}\ \sqrt[3]{2\cdot 5\ xy^2}$

$= -2xyz^2\sqrt[3]{10xy^2}$

II. Rationalizing the Denominator

If there is a fraction under the radical, use the property that

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$ to separate the expression into two radicals.

To eliminate the radical in the denominator, rename the fraction by multiplying the numerator and denominator by the smallest nth root possible that will turn the expression in the denominator into a perfect nth power.

If the denominator of a fraction contains a radical expression with two terms, it is necessary to multiply the numerator and denominator by the **conjugate** of the denominator to rationalize the denominator (see example 5(e)).

Example 2

(a) $$\sqrt{\frac{x}{4y}} = \frac{\sqrt{x}}{\sqrt{4y}} = \frac{\sqrt{x}}{\sqrt{4y}} \cdot \frac{\sqrt{y}}{\sqrt{y}} = \frac{\sqrt{xy}}{\sqrt{4y^2}} = \frac{\sqrt{xy}}{2y}$$

(b) $$\frac{\sqrt[3]{x}}{\sqrt[3]{4y}} = \frac{\sqrt[3]{x}}{\sqrt[3]{2^2y}} = \frac{\sqrt[3]{x}}{\sqrt[3]{2^2y}} \cdot \frac{\sqrt[3]{2y^2}}{\sqrt[3]{2y^2}} = \frac{\sqrt[3]{2xy^2}}{\sqrt[3]{2^3y^3}} = \frac{\sqrt[3]{2xy^2}}{2y}$$

III. Reducing the Index

If the root index and all the exponents of the factors under the radical have a common factor, this factor can be divided out, thereby reducing the index. Be sure to prime-factor the numerical coefficient before applying this technique.

Example 3

(a) $$\sqrt[10]{y^6} = \sqrt[5]{y^3}$$

(b) $$\sqrt[6]{8x^9y^6} = \sqrt[6]{2^3x^9y^6} = \sqrt[2]{2x^3y^2}$$

IV. Adding and Subtracting Radical Terms

To add or subtract radical terms, the terms must be "like" terms. Sometimes, it is necessary to first simplify the radical terms before it becomes clear that they are like terms.

Example 4

(a) $\sqrt{32} + 5\sqrt{18} = \sqrt{16\cdot 2} + 5\cdot\sqrt{9\cdot 2} = 4\sqrt{2} + 5\cdot 3\sqrt{2}$
$= 4\sqrt{2} + 15\sqrt{2} = 19\sqrt{2}$

(b) $\sqrt[3]{54x^4y} - x\sqrt[3]{16xy} = \sqrt[3]{2\cdot 3^3x^4y} - x\sqrt[3]{2^4xy}$
$= \sqrt[3]{3^3x^3}\,\sqrt[3]{2xy} - x\sqrt[3]{2^3}\,\sqrt[3]{2xy}$
$= 3x\sqrt[3]{2xy} - 2x\sqrt[3]{2xy} = x\sqrt[3]{2xy}$

V. Multiplying Radical Factors

To multiply radical factors with the same index, we multiply their radicands and then simplify using the above techniques if possible. When multiplying expressions containing radical terms, we can also use the distributive property and multiplication formulas that were previously used to square binomials and multiply binomials.

Example 5

(a) $\sqrt{14x}\cdot\sqrt{7xy} = \sqrt{2\cdot 7^2x^2y} = \sqrt{7^2x^2}\cdot\sqrt{2y} = 7x\sqrt{2y}$

(b) $4\sqrt[3]{2x^2}\cdot 3\sqrt[3]{12xy^2} = 12\sqrt[3]{24x^3y^2} = 12\sqrt[3]{2^3x^3}\,\sqrt[3]{3y^2}$
$= 12\cdot 2x\sqrt[3]{3y^2} = 24x\sqrt[3]{3y^2}$

(c) $2\sqrt{3}\,(5\sqrt{3} + 4\sqrt{2}) = 2\sqrt{3}\cdot 5\sqrt{3} + 2\sqrt{3}\cdot 4\sqrt{2} = 10\cdot 3 + 8\sqrt{6}$
$= 30 + 8\sqrt{6}$

(d) $(2\sqrt{x} - 3\sqrt{y})^2 = (2\sqrt{x})^2 - 2(2\sqrt{x})(3\sqrt{y}) + (3\sqrt{y})^2$
$= 4x - 12\sqrt{xy} + 9y$

(e) $\dfrac{2+\sqrt{3}}{4\sqrt{3}-\sqrt{2}} = \dfrac{2+\sqrt{3}}{4\sqrt{3}-\sqrt{2}}\cdot\dfrac{4\sqrt{3}+\sqrt{2}}{4\sqrt{3}+\sqrt{2}} = \dfrac{(2+\sqrt{3})(4\sqrt{3}+\sqrt{2})}{(4\sqrt{3}-\sqrt{2})(4\sqrt{3}+\sqrt{2})}$

$= \dfrac{8\sqrt{3} + 2\sqrt{2} + 4\cdot 3 + \sqrt{6}}{16\cdot 3 + 4\sqrt{6} - 4\sqrt{6} - 2} = \dfrac{8\sqrt{3} + 2\sqrt{2} + 12 + \sqrt{6}}{46}$

Math 20

Radicals - Homework Assignment

Simplify each of the following radicals:

1. $\sqrt[4]{9x^6y^7}$

2. $\sqrt[5]{16x^3y^5}$

3. $\sqrt[8]{3^4}$

4. $\sqrt[9]{y^6}$

5. $\sqrt[4]{x^2}$

6. $\sqrt[6]{y^4(x+y)^2}$

7. $\sqrt[6]{81y^8}$

8. $\sqrt[4]{4x^{10}}$

9. $\sqrt{\frac{1}{2}}$

10. $\sqrt[3]{\frac{1}{2}}$

11. $\sqrt[3]{-\frac{3}{4}}$

12. $\sqrt[4]{\frac{5}{8}}$

13. $\sqrt{\frac{9}{x}}$

14. $\sqrt[3]{\frac{x}{y}}$

15. $\sqrt[4]{\frac{2x}{y^5}}$

16. $\sqrt{\frac{5}{4}-\frac{4}{5}}$

Perform the following operations and simplify your answers:

17. $3\sqrt{72} - 5\sqrt{50}$

18. $3\sqrt{28} - 10\sqrt{63}$

19. $4\sqrt{\frac{1}{5}} + 2\sqrt{125}$

20. $\sqrt{56} - \frac{1}{2}\sqrt{\frac{2}{7}}$

21. $2\sqrt[3]{-54} - \sqrt[3]{128}$

22. $3\sqrt[4]{32} - 5\sqrt[4]{2}$

23. $\sqrt{8x} + \sqrt{72x}$

24. $\sqrt{20xy^2} - 2\sqrt{45x^3}$

25. $2\sqrt[3]{54x^4y} - \sqrt[3]{\frac{xy^4}{4}}$

26. $\sqrt[4]{\frac{x}{y}} - \sqrt[4]{xy^3}$

27. $\sqrt{4 + 4x} + \sqrt{16 + 16x}$

28. $\sqrt{6} \cdot \sqrt{2}$

29. $\sqrt[3]{6} \cdot \sqrt[3]{2}$

30. $\sqrt[4]{8} \cdot \sqrt[4]{2}$

31. $\sqrt{15x} \cdot \sqrt{6x}$

32. $-3\sqrt{8xy} \cdot 4\sqrt{3x^3y^2}$

33. $\sqrt[3]{4x^2y} \cdot \sqrt[3]{6x^2y^3}$

Perform the following operations and simplify your answers:

34. $4\sqrt[4]{27x^3} \cdot 5\sqrt[4]{3x^5}$

35. $2\sqrt{5}\,(4\sqrt{10} - 3\sqrt{5})$

36. $(3 + \sqrt{5})(3 - \sqrt{5})$

37. $(2\sqrt{3} - 4\sqrt{5})(2\sqrt{3} + 4\sqrt{5})$

38. $(1 + \sqrt{2})^2$

39. $(2 - \sqrt{5})^2$

40. $(3 - \sqrt{2})(5 - \sqrt{6})$

41. $(3\sqrt{3} - 2\sqrt{6})(2\sqrt{6} - 3\sqrt{3})$

42. $(\sqrt{x} - \sqrt{y})^2$

43. $(\sqrt{x+y})^2$

44. $\dfrac{\sqrt{24}}{\sqrt{3}}$

45. $\dfrac{\sqrt{10}}{\sqrt{3}}$

46. $\dfrac{\sqrt{18}}{\sqrt{3x}}$

47. $\dfrac{4\sqrt{5x}}{\sqrt{2x^3}}$

48. $\dfrac{\sqrt[3]{-3y}}{\sqrt[3]{48y}}$

49. $\dfrac{\sqrt[4]{7x^2}}{\sqrt[4]{2x}}$

50. $\dfrac{\sqrt[5]{5y^4}}{\sqrt[5]{3y^2}}$

51. $\dfrac{-4}{3\sqrt{12}}$

52. $\dfrac{1}{\sqrt[3]{2}}$

53. $\dfrac{3}{1 + \sqrt{2}}$

54. $\dfrac{5 + 2\sqrt{7}}{2 - 2\sqrt{7}}$

Perform the following operations and simplify your answers:

55. $\dfrac{\sqrt{x}}{\sqrt{x} + \sqrt{y}}$

56. $\dfrac{a^2}{\sqrt{a^2 + b^2}} + \dfrac{b^2}{\sqrt{a^2 + b^2}}$

57. $\sqrt{R^2 - x^2} + (R + x)\dfrac{-x}{\sqrt{R^2 - x^2}}$

58. $\dfrac{\dfrac{x^2}{\sqrt{x^2 + 1}} - \sqrt{x^2 + 1}}{x^2}$

59. $-\dfrac{1}{2}\sqrt{\dfrac{x}{y}} \cdot \dfrac{x\sqrt{\dfrac{y}{x}} - y}{x^2}$

60. If $f(x) = x^2 - 2x + 5$, find $f(1 + \sqrt{2})$

61. Find the area of the right triangle with vertices at $(1,1)$, $(-3,2)$, and $(-4,-2)$.

MATH 20
EQUATIONS

Solve each of the following. Circle the final answer. Show your work.

1. $\dfrac{1}{x-1} - \dfrac{1}{2x-2} = \dfrac{1}{2x-2}$

2. $9x^2 + 6x = 1$

3. $12x^2 + x\sqrt{6} - 1 = 0$

4. $(x^2 - x)^2 - 18(x^2 - x) + 72 = 0$

5. $\dfrac{1}{y} = \dfrac{3}{\sqrt{4y+1}}$

6. $\sqrt{x+4} + \sqrt{x-1} = 5$

7. $(3m+1)^{-3/5} = -\dfrac{1}{8}$

8. $-3|x+8|+4 = 1$

9. $\dfrac{1}{(5x-1)^2} + \dfrac{1}{5x-1} - 12 = 0$

10. $x^4 + 6x^2 - 27 = 0$

Solve each of the following equations in $[0°,360°)$. Approximate to the nearest degree if necessary.

11. $-2 \sin x = \tan x$

12. $2 \tan^2 x - \cos^2 x = 1 + \sin^2 x$

13. $\dfrac{2 \tan x}{3 - \tan^2 x} = 1$

14. $12 \cos y + 5 = 2 \sec y$

Solve each of the following equations in $[0,2\pi)$. Approximate to the nearest hundredth if necessary.

15. $\tan \frac{1}{3}\theta = 1$

16. $\cos^2 x - \sin^2 x = 0$

17. $2 \cos^2 \frac{x}{2} - 1 = \cos x$

18. $6 \cos^2 z - \sin^2 z = 1$

Find **all** solutions (in radians) to each of the following. Approximate to the nearest hundredth if necessary.

19. $\sin 2t = \sqrt{2} \cos t$

20. $2 \cos^2 x - 3 \cos x = 2$

ABSOLUTE VALUE INEQUALITIES

Solve each inequality and sketch the graph of the solution on a number line.

1. $|x-3| \leq 5$

2. $|2x+3| \geq 5$

3. $|4x-9| < 1$

4. $|5-3x| > 2$

5. $|9x+15| \geq 30$

6. $\dfrac{|x-3|}{5} \leq 2$

7. $|4-x| \leq 0$

8. $|4-x| > 0$

9. $|x+5| \leq -3$

10. $|x+5| \geq -5$

11. $3|x-2| \leq 15$

12. $-5|x-3| > -10$

13. $\dfrac{1}{2}|3x+1| \geq 7$

14. $-|x-3| \leq 3$

15. $|x-3|-4 \leq 5$

16. $|3x-2| + 7 > 19$

17. $\left|\dfrac{x}{8}+1\right| - 6 \leq 0$

18. $\dfrac{|x+6|}{2} - 3 \geq 5$

19. $1 - 3|7-x| \leq -11$

20. $\dfrac{2}{3}|5x+1| - 1 > 5$

SOLVING POLYNOMIAL INEQUALITIES

Objectives:

1) Solve polynomial inequalities algebraically (using a sign chart)
2) Solve polynomial inequalities using a graph

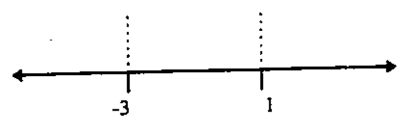
Solving polynomial inequalities hinge on finding the zeros. The zeros are called critical numbers. These numbers separate the number line into intervals and then the sign chart is used to determine the intervals in which the polynomial is either positive or negative.

Example 1: Solve $x^2 + 2x - 3 \geq 0$.

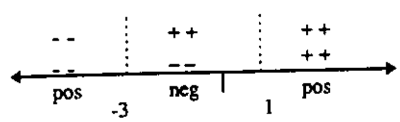
Factor to find the zeros $\qquad (x+3)(x-1) \geq 0$

hence the zeros are -3 and 1.

Place the zeros -3 and 1 on the number line



Arbitrarily choose a value in each interval and determine whether + or -.



The solution includes the critical numbers because it has the ≥ symbol, hence the solution is the intervals (-∞, -3] and [1, ∞).

Example 2: Solve $x^3 < 9x$.

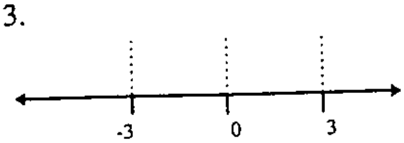
Factor to find the zeros

$$x^3 - 9x < 0$$

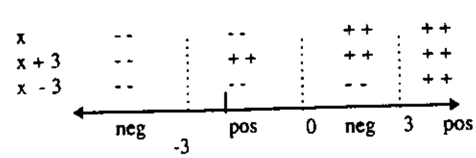
$$x(x+3)(x-3) < 0$$

hence the zeros are 0, -3, and 3.

Place the zeros -3, 0 and 3 on the number line



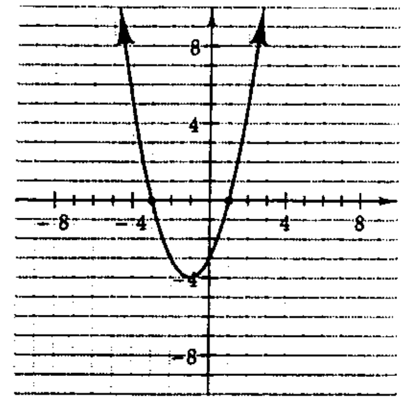
Arbitrarily choose a value in each interval and determine whether + or -.



The solution does **not include the critical numbers** because there is not equal symbol, hence the solution is the intervals (-∞, -3) and (0, 3).

SOLVING USING THE GRAPH

Example 3: Solve $x^2 + 2x - 3 \geq 0$ using its graph.

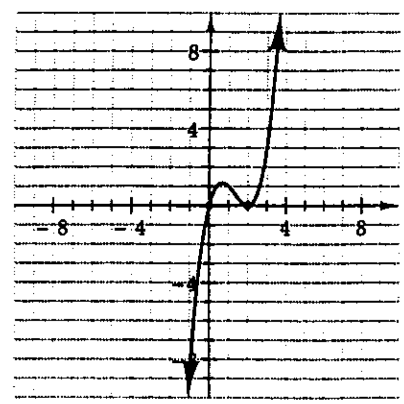


The graph of $y = x^2 + 2x - 3$ is at the right.

The inequalities asks for y values greater than or equal to zero, so the solution would be the interval(s) along the x-axis for which the graph is above the x-axis.

The solution is the intervals $(-\infty, -3]$ and $[1, \infty)$.
It includes the values $f(x) = 0$, the x intercepts.

Example 4: Solve $x(x-2)^2 < 0$.



The graph of $y = x(x-2)^2$ is at the right.

The inequality asks for y values that are less than zero, so the solution would be the x interval for which the graph is below the x-axis.

The solution is the interval $(-\infty, 0)$.
Why does the solution not include zero?

Polynomial Inequality Handout

Solve each of the following graphically. Write your answer in interval notation.

1. $x^2 - 2x \geq 0$
2. $25 - x^2 \geq 0$
3. $(x+3)^2 \leq 0$
4. $x(x-3)^2 \geq 0$
5. $x^2(4-x)(x+6) < 0$
6. $x^2 + 1 \leq 0$

Solve each of the following algebraically. Write your answer in interval notation.

7. $x^2 - 4x \geq 5$
8. $x^2 + 5 \leq 6x$
9. $x^4 < 4x^2$
10. $4x^2 > 12x - 9$
11. $2x^2 - 3 \leq x$
12. $7 - x^2 \leq 0$
13. $x^4 - 5x^2 \leq -4$
14. $x^5 + 9x \geq 10x^3$
15. $x^2 + 9 \leq 0$
16. $2x^3 - 5x^2 < 2x - 5$

SOLVING RATIONAL INEQUALITIES

Objectives:

1) Solve rational inequalities algebraically (using a sign chart)
2) Solve rational inequalities using a base graph

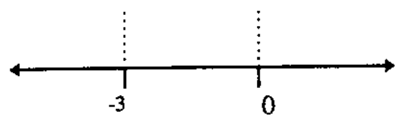
In order to solve rational inequalities it is necessary to find both the zeros and where the inequality is undefined.

Example 1: Solve $\frac{6}{x} + 2 \geq 0$.

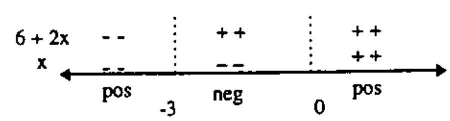
Simplify the left side by finding a common denominator. $\frac{6+2x}{x} \geq 0$.

The zero from the numerator is $x = -3$ and inequality is undefined at $x = 0$. These are called critical numbers.

Place the critical numbers -3 and 0 on the number line



Arbitrarily choose a value in each interval and determine whether + or -.



The solution includes the only one of the critical numbers. The value $x = 0$ cannot be part of the solution because the inequality is not defined there. Hence the solution is the intervals $(-\infty, -3]$ and $(0, \infty)$.

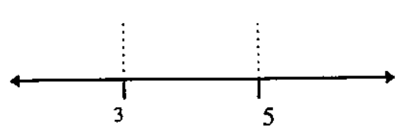
Example 2: Solve $\frac{4}{x-3} < 2$.

First step is to set the inequality to zero. $\frac{4}{x-3} - 2 < 0$.

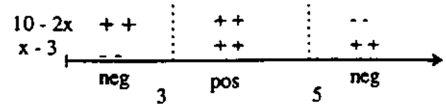
Simplify the left side by finding a common denominator. $\frac{10-2x}{x-3} < 0$.

The zero from the numerator is $x = 5$ and the inequality is undefined at $x = 3$.

Place the critical numbers 5 and 3 on the number line



Arbitrarily choose a value in each interval and determine whether + or -.

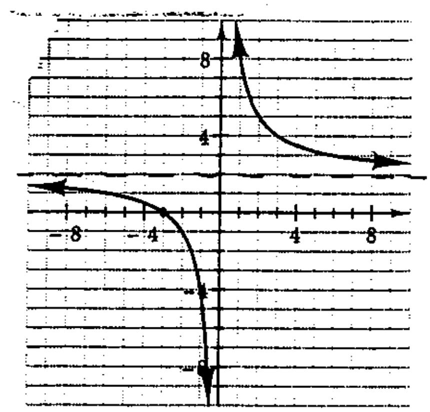


The solution includes none of the critical numbers since there is not equal sign. Hence the solution is the intervals (-∞, 3) and (5, ∞).

SOLVE USING THE BASE GRAPH.

Example 3: Solve $\frac{6}{x} + 2 \geq 0.$

Graph the base graph $y = \frac{1}{x}$ with a horizontal translation of 2 units. See the graph at the right.

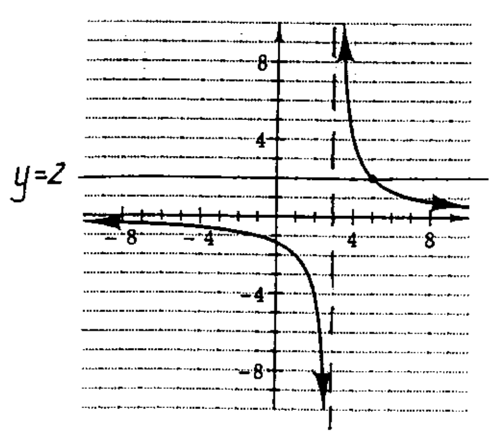


For what interval(s) along the x-axis is the graph above zero, meaning above the x-axis?

Answer: (-∞, -3] and (0, ∞).

Example 4: Solve $\frac{4}{x-3} < 2.$

Graph the base graph $y = \frac{1}{x}$ with a vertical translation of 3 units to the right. See the graph at the right.



For what interval(s) along the x-axis is the graph below the line $y = 2$.

Answer: (-∞, 3) and (5, ∞).

Rational Inequality Handout

Solve each of the following graphically. Write your answer in interval notation.

1. $\frac{1}{x} + 1 \geq 0$

2. $\frac{1}{x^2} - 1 \leq 0$

3. $\frac{3}{x+1} \leq 3$

4. $\frac{-1}{(x-2)^2} \leq 1$

Solve each of the following algebraically. Write your answer in interval notation.

5. $\frac{6}{x} < 2$

6. $\frac{x+3}{x-1} \geq 0$

7. $\frac{x(4-x)}{x+2} \geq 0$

8. $\frac{(x+3)^2}{x} \leq 0$

9. $\frac{x^2-4}{3-x} \geq 0$

10. $\frac{-5}{(x+3)^2} > 0$

11. $\frac{2x}{x-2} \leq 3$

12. $\frac{1}{x+2} \geq \frac{1}{3}$

13. $\frac{1}{4} < \frac{7}{7-x}$

14. $\frac{x+2}{x+5} \geq 1$

15. $\frac{3}{x-2} \leq \frac{3}{x+3}$

16. $x - \frac{10}{x-1} \geq 4$

17. What is the domain of the each of the following functions.

a) $y = \sqrt{x^2 - 4}$

b) $y = \sqrt{\frac{x+1}{x-2}}$

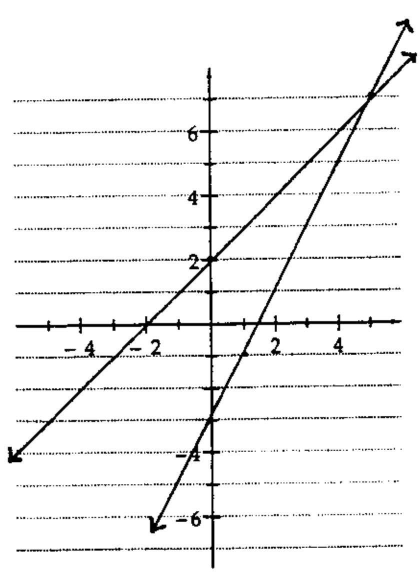
Solving Systems of Linear Equations

I. GRAPHICAL METHOD

When solving a system with two linear equations in two variables, we are looking for the point where the two lines cross. This can be determined by graphing each line on the same coordinate system and estimating the point of intersection. Sometimes, the lines do not cross in which case they are parallel. A system consisting of two **parallel lines** is said to be **inconsistent** and has **no solutions**. Other times, the **two lines coincide** and any point on the line will be a solution to the system. This type of system is said to be **dependent** and has an **infinite number of solutions**. When two lines cross in **exactly one point**, the system is **consistent and independent** and the solution is the one ordered pair where the two lines cross. The coordinates of this ordered pair can be estimated from the graph of the two lines. These three cases are illustrated below. The graphical method is good because it clearly illustrates the principle involved. However, it takes a lot of time, does not always give us an **exact** solution, and cannot be used when we have more than two variables in the equations. When we want exact solutions or want to solve systems with more than two equations in two variables, we must use algebraic methods described in the next section.

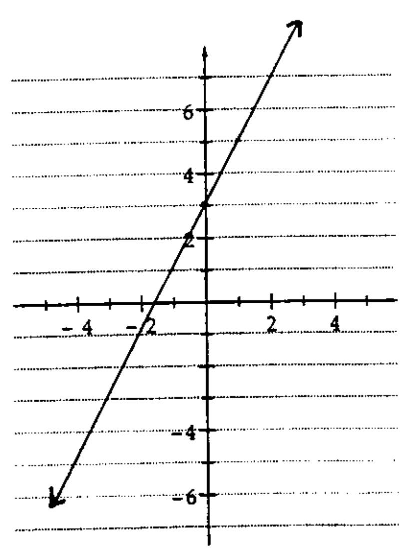
<u>Case 1</u>: Consistent & Independent System

$y = x + 2$
$2x - y = 3$



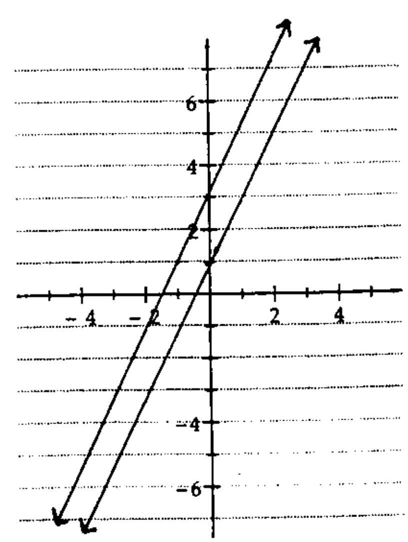
<u>Case 2</u>: Dependent System

$y = 2x + 3$
$2x - y = -3$



<u>Case 3</u>: Inconsistent System

$y = 2x + 1$
$2x - y = -3$



II. ALGEBRAIC METHODS

Recall from algebra that there are two basic algebraic methods of solving linear systems of equations: the **substitution method** and the **elimination** or **addition method**. The goal in each case is to end up with **one equation in one variable.**

To solve the system illustrated in case 1 above using the **substitution method,** we would replace the y in the second equation with the expression x + 2 from the first equation: $2x - (x + 2) = 3$. This is now easy to solve for x:

$$2x - x - 2 = 3$$
$$x - 2 = 3$$
$$x = 5$$

Thus, the x-coordinate of the solution is 5. To find the y-coordinate, we substitute 5 in for x in either of the original equations. It is easier to use the first equation:

$$y = 5 + 2$$
$$y = 7$$

So, the solution to the system is the ordered pair (5,7). This is the point where both lines cross as we saw in section I.

To solve the system illustrated in case 2 above using the **substitution method,** we would replace the y in the second equation with the expression 2x + 3 from the first equation: $2x - (2x + 3) = -3$

$$2x - 2x - 3 = -3$$
$$-3 = -3$$

This gives us a true statement which means that the system is **<u>Dependent</u>**. There are an **<u>infinite number of solutions</u>**.

To solve the system illustrated in case 3 above using the **substitution method,** we would replace the y in the second equation with the expression 2x + 1 from the first equation: $2x - (2x + 1) = -3$

$$2x - 2x - 1 = -3$$
$$-1 = -3$$

This gives us a false statement which means that the system is **<u>Inconsistent</u>**. There are **<u>no solutions</u>**.

Many systems are easier to solve using the **elimination or addition method.** This is especially true when we have a system with more than two variables. We can also avoid fractions when using this method. In the elimination/addition method, we multiply one or both of the equations by a constant so that when we add them together, we eliminate one of the variables. Remember that multiplying an equation by a constant produces an equivalent equation. The goal is to eventually end up with one equation in one variable.

Example: Solve the following system using the **elimination/addition method**:

$$\begin{cases} 3x - 2y = 27 & (1) \\ 2x + 5y = -1 & (2) \end{cases}$$

To eliminate y, we could multiply the first equation by 5 and the second equation by 2, and add them together:

$$\begin{array}{rll} & 5\,(1): & 15x - 10y = 135 \\ + & \underline{2\,(2):} & \underline{\;\;4x + 10y = -2} \\ & & 19x \qquad\quad = 133 \\ & & \quad x = 7 \end{array}$$

To find the y-coordinate, we substitute in 7 for x in either of the original two equations:

$$\begin{aligned} 3(7) - 2y &= 27 \\ 21 - 2y &= 27 \\ -2y &= 6 \\ y &= -3 \end{aligned}$$

Thus the solution is $(7,-3)$.

Example: Solve the following system using the **elimination/addition method**:

$$\begin{cases} 2x - 4y + 3z = 31 & (1) \\ 5x - 2y - 2z = 6 & (2) \\ 3x + 4y + 5z = 19 & (3) \end{cases}$$

We must choose a variable to eliminate, and eliminate it using two **different** pairs of equations. It appears that y is the easiest variable to eliminate.

$$\begin{array}{rll} & (1): & 2x - 4y + 3z = 31 \\ + & \underline{-2\,(2):} & \underline{-10x + 4y + 4z = -12} \\ & (4): & -8x \qquad + 7z = 19 \end{array} \qquad \begin{array}{rll} & (1): & 2x - 4y + 3z = 31 \\ + & \underline{(3):} & \underline{3x + 4y + 5z = 19} \\ & (5): & 5x \qquad + 8z = 50 \end{array}$$

Equations (4) and (5) now form a reduced system with only two equations in two variables. We solve it in a similar manner by eliminating x:

$$\begin{array}{rll} & 5\,(4): & -40x + 35z = 95 \\ + & \underline{8\,(5):} & \underline{\;\;40x + 64z = 400} \\ & & \qquad 99z = 495 \\ & & \qquad\; z = 5 \end{array}$$

We go back to our reduced system to now find x using equation (5):

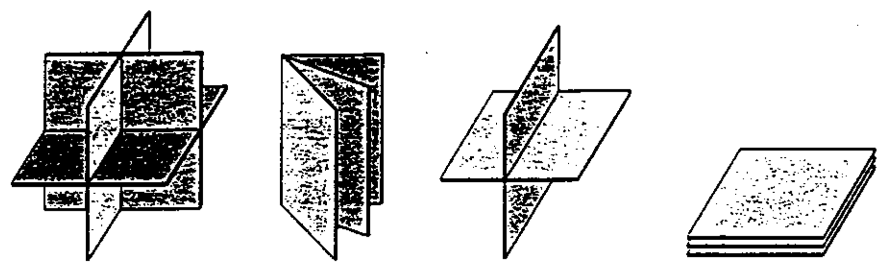
$$\begin{aligned} 5x + 8(5) &= 50 \\ 5x + 40 &= 50 \\ 5x &= 10 \\ x &= 2 \end{aligned}$$

We can find the remaining variable y by substituting both $z = 5$ and $x = 2$ into one of the original equations:

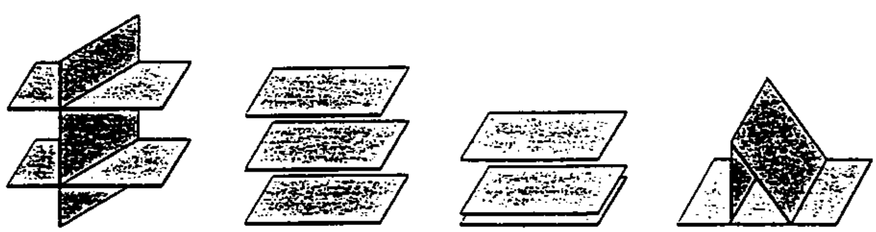
$$\begin{aligned} (1): \quad 2(2) - 4y + 3(5) &= 31 \\ 4 - 4y + 15 &= 31 \\ -4y + 19 &= 31 \\ -4y &= 12 \\ y &= -3 \end{aligned}$$

Thus the ordered triple $(x,y,z) = (2,-3,5)$ is the solution to the system.

Note that the solution to the last example was an ordered triple which is a point in space using a three-dimensional Cartesian coordinate system. In this coordinate system, the graph of a linear equation in three variables is a plane. The fact that we obtained just one ordered triple as a solution to the last example indicates that the three planes intersected in a single point. (See figure (a) below.) There are many other possibilities, as illustrated below. In figures (b), (c), and (d) the intersection is either a line or an entire plane, so the corresponding system has infinitely many solutions. As before, a system like this is called **dependent**. In figures (e), (f), (g), and (h) the three planes have no common intersection, so the corresponding system has no solution. A system like this is called **inconsistent**. Algebraically, we recognize a dependent system if at any step in the process we eliminate all the variables but end up with a true statement (like $0 = 0$). If at any step in the process we eliminate all the variables but end up with a false statement (like $0 = 7$) then the system is **inconsistent**.



(a) (b) (c) (d)



(e) (f) (g) (h)

Example: Solve the following system using the **elimination/addition method**:

$$\begin{cases} 2x + 3y - z = 10 & (1) \\ 3x - 2y + z = 7 & (2) \\ -4x - 6y + 2z = 9 & (3) \end{cases}$$

Eliminate z using two different pairs of equations:

$$\begin{array}{lll} & (1): & 2x + 3y - z = 10 \\ + & (2): & 3x - 2y + z = 7 \\ \hline & (4): & 5x + y \quad = 17 \end{array} \qquad \begin{array}{ll} -2\,(2): & -6x + 4y - 2z = -14 \\ (3): & -4x - 6y + 2z = 9 \\ \hline (5): & -10x - 2y \quad = -5 \end{array}$$

Eliminate y using equations (4) and (5) by first multiplying equation (4) by 2 and then adding it to equation (5):

$$\begin{array}{ll} 2\,(4): & 10x + 2y = 34 \\ (5): & -10x - 2y = -5 \\ \hline & 0 = 29 \end{array}$$

Since this is a false statement, the system is **Inconsistent** and has **no solution**.

SYSTEMS OF EQUATIONS

Find all solutions that satisfy the system using either elimination or substitution. If there is no solution or infinitely many solutions, state so.

1. $\begin{aligned} 2x + y &= 3 \\ 3x + 5y &= 1 \end{aligned}$

2. $\begin{aligned} 2x - 2y &= 0 \\ x &= y - 1 \end{aligned}$

3. $\begin{aligned} -2x + 6y &= 3 \\ 4x - 12y &= -6 \end{aligned}$

4. $\begin{aligned} \frac{2}{3}x - y &= 0 \\ 10x + 4y &= 19 \end{aligned}$

5. $\begin{aligned} x - 2 &= 3y \\ 3x - 7y &= 4 \end{aligned}$

6. $\begin{aligned} 6y &= 13x + 17 \\ 26x - 12y &= 8 \end{aligned}$

7. $\begin{aligned} \frac{x}{3} + \frac{y}{7} &= 2 \\ \frac{x}{6} - \frac{y}{3} &= 1 \end{aligned}$

8. $\begin{aligned} \frac{x-3}{2} &= \frac{y-5}{4} \\ \frac{x+5}{2} &= \frac{2y+7}{5} \end{aligned}$

9. $\begin{aligned} x + y + z &= -3 \\ 4x + y - 3z &= 11 \\ 2x - 3y + 2z &= 9 \end{aligned}$

10. $\begin{aligned} 2x + 2z &= 2 \\ 5x + 3y &= 4 \\ 3y - 4z &= 4 \end{aligned}$

11. $\begin{aligned} x + 2y + 6z &= 5 \\ -x + y - 2z &= 3 \\ x - 4y - 2z &= 1 \end{aligned}$

12. $\begin{aligned} 2x - 3y + 3z &= 5 \\ x - 3z &= 1 \\ 4x - 6y + 6z &= 10 \end{aligned}$

13. $\begin{aligned} 2x + y - 2z &= 4 \\ 3x - 2y + 4z &= 6 \\ -4x + y + 6z &= 12 \end{aligned}$

14. $\begin{aligned} 2x + 4z &= 1 \\ x + y + 3z &= 0 \\ x + 3y + 5z &= 0 \end{aligned}$

15. $\begin{aligned} 2x - 4y + z &= 0 \\ 3x + 2z &= -1 \\ -6x + 3y + 2z &= -10 \end{aligned}$

16. $\begin{aligned} x + 4y - 2z &= 2 \\ x + y + z &= -1 \\ 5x + 7y + 3z &= -3 \end{aligned}$

Solving Systems of Linear Equations Using Matrices

I. Gauss–Jordan Method

A very systematic method of solving linear systems of equations is called the Gauss-Jordan Method. This method involves the use of matrices, the plural of the word matrix. A **matrix** is a rectangular array of numbers arranged in rows and columns. The numbers in the array are called the elements of the matrix.

To solve linear systems of equations, we will first form the **augmented matrix** by writing the numerical coefficients and constants of each equation in a matrix. Before forming the matrix, be sure to line-up the variables so that they are in the same column; any missing terms will require a coefficient of 0. We separate the coefficients from the constants with a vertical line. Two examples are:

System of Linear Equations	Augmented Matrix
$\begin{cases} 5x + 8y = 2 \\ 7x - 3y = 88 \end{cases}$	$\left[\begin{array}{cc\|c} 5 & 8 & 2 \\ 7 & -3 & 88 \end{array}\right]$
$\begin{cases} x + 2y + 4z = 24 \\ 2x - y - 3z = -22 \\ 3x + 5z = 19 \end{cases}$	$\left[\begin{array}{ccc\|c} 1 & 2 & 4 & 24 \\ 2 & -1 & -3 & -22 \\ 3 & 0 & 5 & 19 \end{array}\right]$

If two systems of linear equations have the same solution sets, then the systems are said to be equivalent. Similarly, the augmented matrices of equivalent systems are also equivalent.

Example: Verify that the augmented matrices A and B are equivalent:

$$A = \left[\begin{array}{ccc|c} 1 & 6 & -2 & -14 \\ 3 & 0 & 2 & 4 \\ 5 & -3 & 3 & 1 \end{array}\right] \qquad B = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

By definition, the two augmented matrices are equivalent if the associated system of linear equations have the same solution sets. The solution set of the system associated with matrix B can be readily identified by converting back to equations: $\begin{cases} x = -4 \\ y = 1 \\ z = 8 \end{cases}$

We must show that this is also a solution to the system associated with matrix A:

$$\begin{cases} x + 6y - 2z = -14 \\ 3x \quad\quad + 2z = 4 \\ 5x - 3y + 3z = 1 \end{cases}$$

Substituting in -4 for x, 1 for y, and 8 for z produces a true statement in each case (verify this!) so the matrices are indeed equivalent.

In the **Gauss–Jordan Method**, our goal is to transform our original augmented matrix into one like matrix B in the above example, with ones along the main diagonal from upper left to lower right and zeros everywhere else. Then the solution will be readily identified. To do this, we use **elementary row operations**, which produce equivalent matrices. Each row operation corresponds to an operation which can be performed on a system of equations to produce an equivalent system.

Elementary Row Operations

1. Any two rows of the matrix may be interchanged. (This corresponds to interchanging the position of any two equations in a system.)

2. The elements of any row of the matrix may be multiplied or divided by a nonzero constant. (This corresponds to multiplying or dividing both sides of an equation by the same nonzero constant.)

3. A nonzero multiple of the elements of any row may be added to the corresponding elements in any other row of the matrix. (This is what we do when we use the **elimination/addition method**!)

A definite strategy must be used to transform the augmented matrix into the form required. The following procedure will always work.

1. Work column by column, left to right.
2. Within a given column, get the "one" in the correct position first. This can be done by dividing the row by the entry or interchanging two rows. As you progress through this procedure, you will only interchange a given row with one **below** it.
3. Use the row with the "one" in it to get the "zeros". This is done using row operation 3. You will multiply the row with the "one" in it by the opposite of the entry that you wish to make zero. Then, when you add the rows together, you will get the zero in that position. Note that you do not actually **change** the row with the "one" in it in the matrix; it is only **used** to get the zero.

Example: Solve the following system using the **Gauss-Jordan Method**:

$$\begin{cases} 4x + y - 3z = 11 & (1) \\ 2x - 3y + 2z = 9 & (2) \\ x + y + z = -3 & (3) \end{cases}$$

The augmented matrix is:

$$\left[\begin{array}{ccc|c} 4 & 1 & -3 & 11 \\ 2 & -3 & 2 & 9 \\ 1 & 1 & 1 & -3 \end{array}\right]$$

Start with column 1. To get a one in the top position, we may interchange rows 1 and 3 to get the following matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 2 & -3 & 2 & 9 \\ 4 & 1 & -3 & 11 \end{array}\right]$$

Now we will use row 1 to get the zeros in rows 2 and 3. To get the zero in row 2, we multiply row 1 by − 2 and add it to row 2:

	− 2 (row 1):	− 2	− 2	− 2	6
+	old row 2:	2	− 3	2	9
	new row 2:	0	− 5	0	15

To get the zero in row 3, we multiply row 1 by − 4 and add it to row 3:

	− 4 (row 1):	− 4	− 4	− 4	12
+	old row 3:	4	1	− 3	11
	new row 3:	0	− 3	− 7	23

Our augmented matrix now looks as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & -5 & 0 & 15 \\ 0 & -3 & -7 & 23 \end{array}\right]$$

The next step is to move to column 2. We first get the one in the proper position by dividing row 2 by − 5: New row 2: 0 1 0 − 3

Our augmented matrix now looks as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & 1 & 0 & -3 \\ 0 & -3 & -7 & 23 \end{array}\right]$$

We will now use row 2 to get the required zeros. To get the zero in row 1, we multiply row 2 by − 1 and add it to row 1:

	− 1 (row 2):	0	− 1	0	3
+	old row 1:	1	1	1	− 3
	new row 1:	1	0	1	0

To get the zero in row 3, we multiply row 2 by 3 and add it to row 3:

	3 (row 1):	0	3	0	− 9
+	old row 3:	0	− 3	− 7	23
	new row 3:	0	0	− 7	14

Our augmented matrix now looks as follows:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & -7 & 14 \end{array}\right]$$

To finish, we look at column 3 and get the one by dividing the entries by – 7:
New row 3: 0 0 1 – 2

Our augmented matrix now looks as follows:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

We will now use row 3 to get the required zeros. To get the zero in row 1, we multiply row 3 by – 1 and add it to row 1:

	– 1 (row 3):	0	0	– 1	2
+	old row 1:	1	0	1	0
	new row 1:	1	0	0	2

Row 2 already has a zero in the third position so we are finished! Our final augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

This means that the solution to the system is $(x,y,z) = (2,-3,-2)$.

The **Gauss–Jordan Method** is a very systematic method for solving systems and it works especially well with large systems. Sometimes you end up with fractional entries which makes computations more difficult, but overall it is a very organized method.

In the process of solving a system of linear equations using the **Gauss–Jordan Method**, two results can happen that indicate that the system is inconsistent or dependent:

1. If the elements of any row are all zeros to the left of the vertical bar and a nonzero constant to the right side, then the associated system is inconsistent. (Note that this would correspond to an equation like $0 = 7$)
2. If the elements of any row are all zeros (to the left and right of the vertical bar), then the associated system is dependent.
(Note that this would correspond to the equation $0 = 0$)

HOMEWORK ASSIGNMENT
GAUSS–JORDAN METHOD

Solve each of the following systems using the Gauss-Jordan Method. Clearly show your steps!

1. $\begin{cases} x - y = 3 \\ -x - y = -3 \end{cases}$

2. $\begin{cases} x - 2y = -5 \\ 2x + 3y = 11 \end{cases}$

3. $\begin{cases} x - 3y = 15 \\ y = \frac{1}{3}x - 5 \end{cases}$

4. $\begin{cases} 3x - 2y = 1 \\ 6x - 4y = 5 \end{cases}$

5. $\begin{cases} x + y = 4 \\ y + z = -8 \\ x + z = 2 \end{cases}$

6. $\begin{cases} x + y + 3z = 1 \\ 2x + 5y + 2z = 0 \\ 3x - 2y - z = 3 \end{cases}$

7. $\begin{cases} 6x + 3y + 2z = 1 \\ 5x + 4y + 3z = 0 \\ x + y + z = 0 \end{cases}$

8. $\begin{cases} x + y - z = -10 \\ 2x + y + z = 2 \\ 3x + 5y - 8z = -66 \end{cases}$

9. $\begin{cases} x + 3y - 2z = 14 \\ 3x - 2y + z = -8 \\ -2x - 6y + 4z = -30 \end{cases}$

10. $\begin{cases} x + y - 3z = -1 \\ y - z = 0 \\ -x + 2y = 1 \end{cases}$

11. $\begin{cases} x + 3w = 4 \\ 2y - z - w = 0 \\ 3y - 2w = 1 \\ 2x - y + 4z = 5 \end{cases}$

APPLICATIONS HOMEWORK

1. Tickets for a barbecue were \$8 for a single ticket or \$15 for a couple. If 144 people attended the barbecue and \$1098 was collected from ticket sales, how many couples and how many singles attended?

2. A boat went 15 miles downstream in 1 hour and then returned upstream to its dock in 5/3 hours. If the current of the stream was constant, find the speed of the boat in still water and the speed of the current.

3. Two different routes between two cities differ by 20 miles. Bill and Lou made the trip between the cities in exactly the same time. If one traveled the shorter route at 50 mph and the other traveled the longer route at 55 mph, how long is each route?

4. The perimeter of a triangle is 54 centimeters. Find the lengths of the three sides if the longest one is twice as long as the shortest one and the other one is 6 centimeters more than the shortest one.

5. A class of 32 students was made up of people who were all 18, 19 and 20 year olds. The average of their ages was 18.5. How many of each age were in the class if the number of 18-year-old was six more than the combined number of 19- and 20-year-olds?

6. Find the equation of the parabola $y = ax^2 + bx + c$ that passes through the points $(0, -4)$, $(1, 1)$, and $(2, 10)$.

7. Find the equation of the parabola $y = ax^2 + bx + c$ that passes through the points $(1, 2)$, $(2, 1)$, and $(3, -4)$.

8. Find the equation of the circle $x^2 + y^2 + Dx + Ey + F = 0$ that passes through the points (0, 0), (4, 0), and (2, -2).

9. Find the equation of the circle $x^2 + y^2 + Dx + Ey + F = 0$ that passes through the points (3, -1), (-2, 4), and (6, 8).

10. Ron, Kent and Sharon assembled 734 newsletters. Ron could assemble 124 per hour, Kent 118 per hour, and Sharon 132 per hour. One morning the three worked a total of 6 hours. If Ron worked 2 hours, how long did Kent and Sharon work?

11. A mixture of 12 gallons of chemical A, 16 gallons of chemical B, and 26 gallons of chemical C is required to kill a certain destructive crop insect. Commercial spray X contains 1, 2, and 2 parts of chemical A, B, and C respectively. Commercial spray Y contains only chemical C. Commercial spray Z contains only chemical A and B in equal amounts. How much of each type of commercial spray is needed to obtain the desired mixture?

12. Oscar invests \$20,000 in three investments earning 6%, 8% and 10% per year. He invests \$9000 more in the 10% investment than in the 6% investment. How much does he have invested at each rate if he receives \$1780 interest the first year?

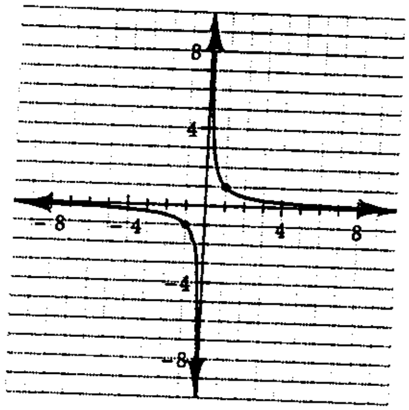
13. Stewart's Metals has three silver alloys on hand. One is 22% silver, another is 30% silver, and the third is 42% silver. How many grams of each alloy is required to produce 80 grams of a new alloy that is 34% silver if the amount of 30% alloy used is twice the amount of 22% alloy used?

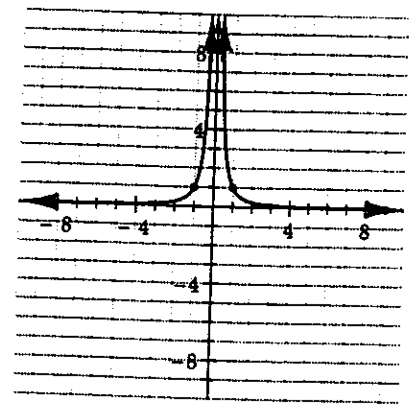
GRAPHS OF RATIONAL FUNCTIONS

Objectives:

1) find any vertical asymptotes and the behavior about them;
2) find any horizontal asymptote;
3) find any oblique asymptote.

A rational function is in the form $y = \dfrac{P(x)}{Q(x)}$ *where* $Q(x) \neq 0$ *and* $P(x)$ *and* $Q(x)$ are polynomials. You are already familiar with the rational functions $y = \dfrac{1}{x}$ *and* $y = \dfrac{1}{x^2}$ and their graphs (see below).





If $P(x)$ and $Q(x)$ have no common factors, then the graph of the rational function $y = P(x)/Q(x)$ has as vertical asymptote the line $x = a$ for each value of a at which $Q(a) = 0$.

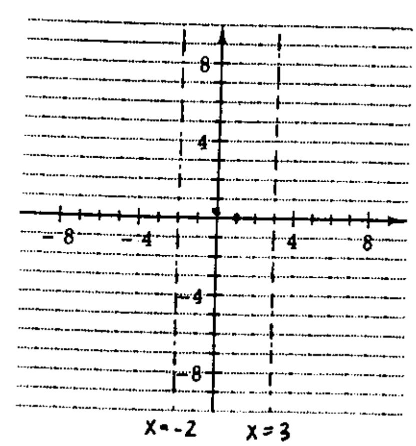
Example 1: Find the vertical asymptotes and the intercepts of the function $y = \dfrac{x-1}{x^2 - x - 6}$.

The function is undefined when $x^2 - x - 6 = 0$.

$$(x-3)(x+2) = 0$$

$$x = 3 \text{ or } -2$$

Hence the vertical asymptotes are at ***x = 3 and x = -2.***



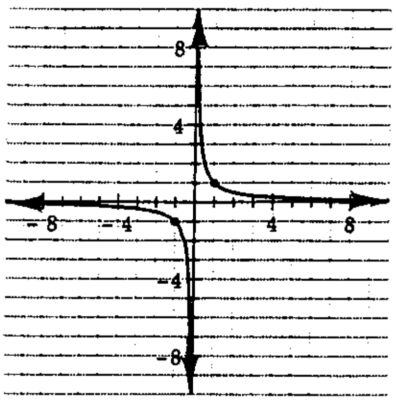
The y-intercept is (0, 1/6) and the x-intercept is (1, 0). The vertical asymptotes and the intercepts are shown on the graph at the right.

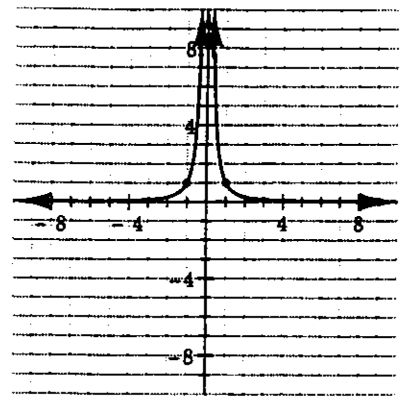
If $(x-a)^n$ is a factor of $Q(x)$ then,

a) the graph of $f(x)$ goes in opposite directions about the vertical asymptote $x = a$ when n is odd,

b) the graph of $f(x)$ goes in the same direction about the vertical asymptote $x = a$ when n is even.

This is illustrated by the base graphs of $y = \frac{1}{x}$ *and* $y = \frac{1}{x^2}$.



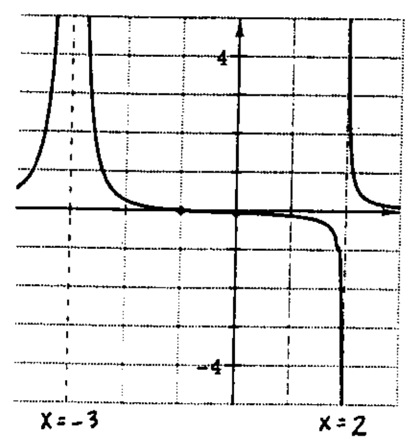


Example 2: Show the intercepts and vertical asymptotes and the behavior of f(x) near the asymptotes for $f(x) = \dfrac{x+1}{(x-2)(x+3)^2}$.

The x-intercept is $(-1, 0)$.

The y-intercept is $\left(0, -\frac{1}{18}\right)$.

The vertical asymptote, $x = 2$, has behavior in the opposite directions and the asymptote, $x = -3$, has behavior in the same direction.



HORIZONTAL ASYMPTOTES

The graph of the rational function

$$y = \frac{P(x)}{Q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ . \ . \ . + a_0}{b_m x^m + b_{m-1}x^{m+1} + \ . \ . \ . + b_0} \quad \textit{where } a_n, \ b_m \neq 0, \text{ has}$$

a) a horizontal asymptote at $y = 0$ if $n < m$

b) a horizontal asymptote at $y = a_n/b_m$ if $n = m$

c) an oblique asymptote if $n > m$.

Example 3: Find the horizontal asymptotes for each of the following:

a) $f(x) = \dfrac{2x-5}{x+3}$ b) $f(x) = \dfrac{x+3}{x^2-1}$

Solution a) The degree of the numerator and denominator is the same, hence the horizontal asymptote is $y = 2/1 = 2$.

Solution b) The degree of the numerator is less than the degree of the denominator, hence the horizontal asymptote is $y = 0$.

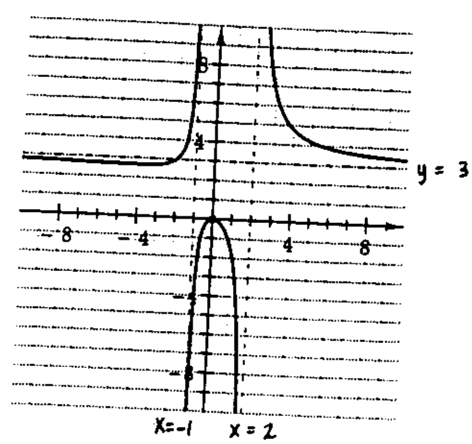
GRAPHING TECHNIQUES:

a) domain
b) x intercept(s) and whether they are bounces or pass throughs (multiplicity)
c) y intercept
d) vertical asymptotes and the behavior around them
e) horizontal asymptote.

Example 4: Sketch the graph of $f(x) = \dfrac{3x^2}{(x-2)(x+1)}$.

The x-intercept is (0, 0) and it is a bounce.
The y-intercept is (0, 0).
The vertical asymptotes are $x = 2$ *and* $x = -1$.
The behavior is in opposite directions.
The horizontal asymptote is $y = 3$.

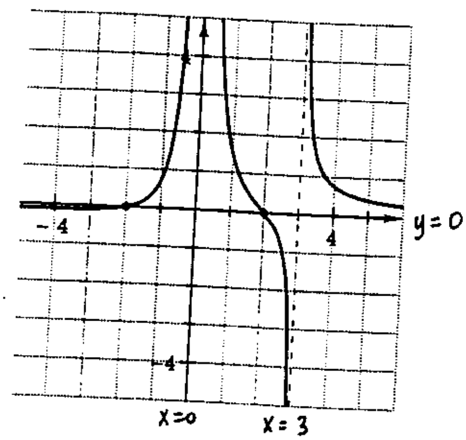
It should be noted that **it is possible to cross the horizontal asymptote**. The graph crosses the horizontal asymptote at $x = -2$.



Example 5: Sketch the graph of $f(x) = \dfrac{x^2 - 4}{x^2(x-3)}$.

The x-intercepts are (-2, 0) and (2, 0) and each is a pass through.
There is no y-intercept.
The vertical asymptotes are $x = 0$ *and* $x = 3$. The behavior about $x = 0$ is in the same direction and the behavior about $x = 3$ is in the opposite directions.
The horizontal asymptote is $y = 0$.

The major decision here is "do you begin the graph **below or above the x-axis**?" Test the end behavior by mentally replacing x by an extremely large negative number. If x is a large negative then y is a small negative number so begin the graph below the x-axis.



HOLES IN THE GRAPH:

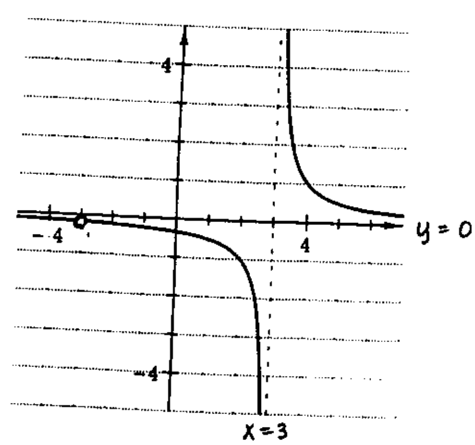
Up to this point all the rational functions have had no common factors. If there is a common factor then divide that factor out but you must keep track of it as it will form a hole in the graph.

Example : Let $y = \dfrac{x+3}{x^2-9}$. Name the vertical asymptotes and any holes.

The domain of the function is all reals except $x = \pm 3$. The common factor is x + 3.

Simplifying the function leads to $y = \dfrac{1}{x-3}$.

The vertical asymptote is at x = 3 and there is a hole in the graph at x = -3.



OBLIQUE ASYMPTOTES:

Oblique asymptotes occur when the degree of the numerator is greater than the degree of the denominator. In primary method for finding the oblique asymptotes is to use long division. **The oblique asymptote will always be the quotient.**

Example 7: Let $f(x) = \dfrac{x^2}{x-2}$. Name the vertical and oblique asymptote.

The domain of the function is all reals except x = 2, hence the vertical asymptote is x = 2.

Long division leads to

$$\begin{array}{r l} & x + 2 \\ x - 2 & \overline{)\, x^2 } \\ & \underline{x^2 - 2x} \\ & \phantom{x^2 -{}} 2x \\ & \phantom{x^2 -{}} \underline{2x - 4} \\ & \phantom{x^2 - 2x -{}} 4 \end{array}$$

So the oblique asymptote is at y = x + 2.

RATIONAL GRAPHS HOMEWORK I

1. For each of the following state the domain, and all vertical and horizontal asymptotes:

   a) $y = \dfrac{4x+1}{3x-6}$ b) $y = \dfrac{x^2-1}{4-x^2}$ c) $y = \dfrac{x+3}{(x-2)^2}$

   d) $y = \dfrac{2x^2+1}{6-5x-x^2}$ e) $y = \dfrac{x}{9-x^2}$ f) $y = \dfrac{5x^2}{(x+1)^2}$

2. For each of the functions in problem #1 state the x and y intercepts, if any.

3. For each of the functions in problems #1 show or discuss the behavior of the graph around the vertical asymptote(s).

4. Sketch a graph for each of the functions in problem #1. Clearly show:
   a) all intercepts
   b) vertical asymptotes
   c) horizontal asymptotes

5. Sketch a graph for each of the following functions. Clearly show:
   a) all intercepts
   b) vertical asymptotes
   c) horizontal asymptotes

   a) $y = \dfrac{4}{x^2-x-2}$ b) $y = \dfrac{4x}{x^2-x-2}$ c) $y = \dfrac{4x^2}{x^2-x-2}$

RATIONAL GRAPHS HANDOUT II

1. For each of the following, determine if there is a hole in the graph and if so where is it.

a) $y = \dfrac{x(x+1)}{2x^2 + 5x + 3}$

b) $y = \dfrac{x^2 - x - 12}{x + 3}$

c) $y = \dfrac{x^2 + x}{x^2 + 5x - 6}$

d) $y = \dfrac{3x^2}{2x^2 + x}$

2. Find the oblique asymptote, if any, for each of the following. Also name the x-intercepts.

a) $y = \dfrac{x^2 - 1}{2x}$

b) $y = \dfrac{x^2 - 3x + 2}{x + 2}$

c) $y = \dfrac{9 - x^2}{(x-4)^2}$

3. Form a complete graph for each of the following. Include all intercepts and asymptotes.

a) $y = \dfrac{x(x+1)}{2x^2 + 5x + 3}$

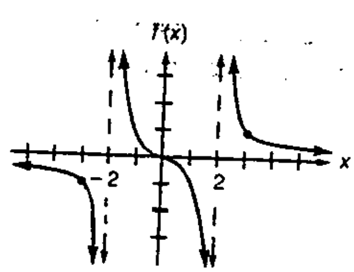
b) $y = \dfrac{x^2 - 3x + 2}{x + 2}$

c) $y = \dfrac{x^2 - 1}{2x}$

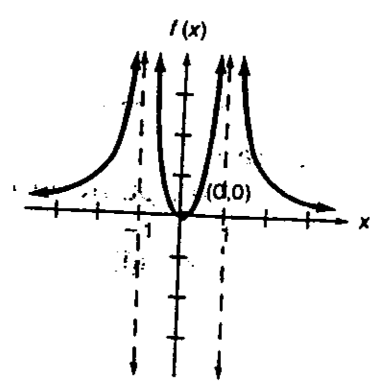
d) $y = \dfrac{(x-2)^2(x+1)}{(x+3)^2(x+1)}$

4. Consider each of the following graphs. Write a rational function that might have that graph.

a)



b)



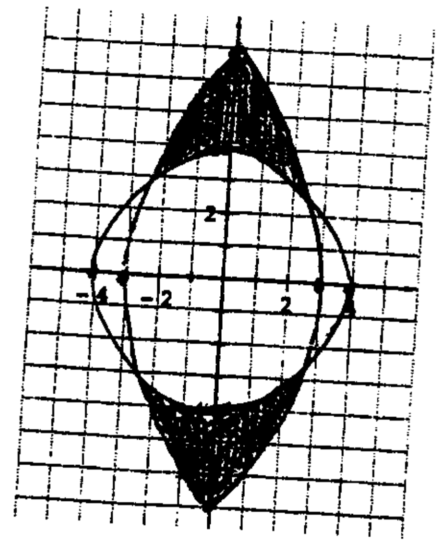
NON-LINEAR SYSTEMS OF INEQUALITIES

Objectives: To graphically display the solution set for a non-linear system of equations. To incorporate all of the different base graphs learned during this course.

Example 1: Sketch the solution set to the system $\begin{array}{l} x^2 + y^2 \geq 16 \\ \dfrac{x^2}{9} + \dfrac{y^2}{49} \leq 1 \end{array}$.

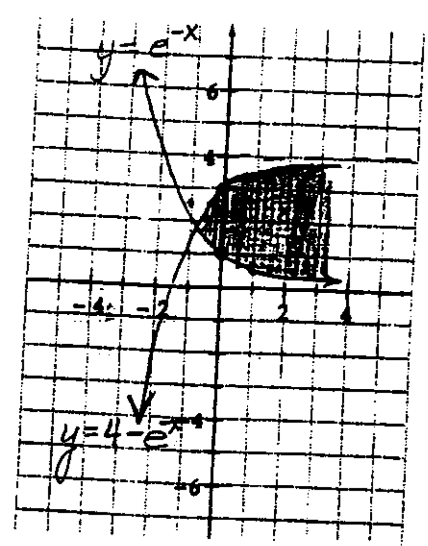
Graph each equation as an equality. The first is a circle whose radius is 4. Choose a point that is not on the circle, say (0, 0), and determine if this point makes the inequality true or false. In this case $0^2 + 0^2 \geq 16$ is not true so the region is outside the circle.
The second is an ellipse with vertices at (0, ±7).

Choose a test point, say (0, 0). Is $\dfrac{0^2}{9} + \dfrac{0^2}{16} \leq 1$?
Since this is true the region is inside the ellipse. The common area is outside the circle but inside the ellipse.



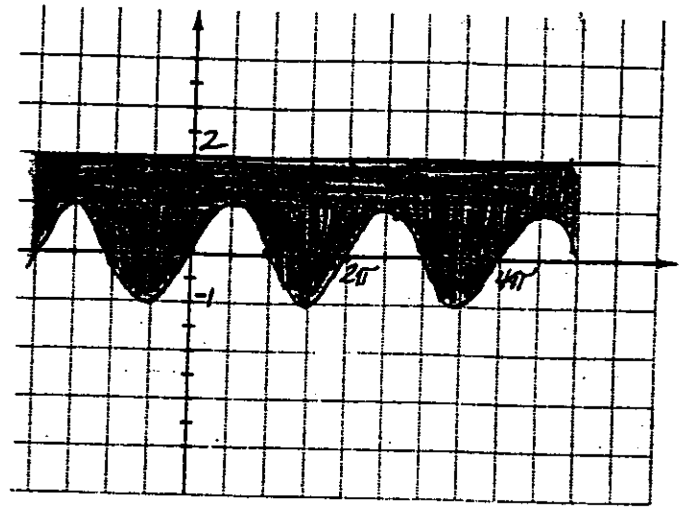
Example 2: Sketch the solution set to the system $\begin{cases} y \geq e^{-x} \\ y \leq 4 - e^{-x} \end{cases}$.

The first equation is the exponential base graph with the base less than one. Choose a test point, say (0, 0). Is $0 \geq e^0$? Since it is not, the region is above the base graph.
The second equation is the same base graph that has been reflected about the x-axis then moves four units up. Choose the test point (0, 0). Is $0 \leq 4 - e^0$? Since it is, the region is below the translated base graph.



Example 3: Sketch the solution set to the system $\begin{cases} y \geq \sin x \\ y \leq 2 \end{cases}$

The first equation is the sine base graph. If we choose the test point (0, 1), notice that $1 \geq \sin 0$ is true. Hence the region is above the base graph.
The second equation is a horizontal line
The region is below the line.
The common region is above the sine graph and below the horizontal line.



Non-Linear Systems of Inequalities

Solve each of the following systems:

1. $\begin{cases} x + 3y \le 3 \\ 2x^2 + y^2 \ge 18 \end{cases}$

2. $\begin{cases} x^2 + y^2 < 13 \\ y \ge x^2 - 1 \end{cases}$

3. $\begin{cases} x^2 - y^2 \ge -5 \\ 3x^2 + 2y^2 \le 30 \end{cases}$

4. $\begin{cases} x^2 + y^2 \ge 3 \\ x \ge y^2 \end{cases}$

5. $\begin{cases} y \le 3^x + 2 \\ y \ge -1 \end{cases}$

6. $\begin{cases} y \ge \left(\frac{1}{4}\right)^x - 1 \\ y \le 4 - x^2 \end{cases}$

7. $\begin{cases} x \le 3 \\ y \ge \log_3 x \\ -2 \le y \le 5 \end{cases}$

8. $\begin{cases} y \ge \log_{1/2} x \\ y < 6 \\ x \le 8 \end{cases}$

9. $\begin{cases} y \ge x \\ y \le \cos x \\ x \ge 0 \end{cases}$

10. $\begin{cases} y \ge -x \\ y \le 1 \\ y \ge \tan x \end{cases}$

11. $\begin{cases} y \ge -\sqrt{x+1} \\ y \le x+1 \\ x \le 3 \end{cases}$

12. $\begin{cases} \dfrac{x^2}{25} + \dfrac{(y-2)^2}{4} \le 1 \\ y \ge \dfrac{1}{x^2} \end{cases}$

13. $\begin{cases} y \le \ln(x+2) \\ y \ge |x| - 1 \end{cases}$

14. $\begin{cases} y \ge x^3 \\ x^2 - y^2 \le 1 \end{cases}$